Question 1 (i)

$$\mathcal{H} = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}k\left(q_x^2 + q_y^2\right), \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\begin{aligned}
\Omega_N &= \frac{1}{h^{2N}}\int_{E<H<E+\delta E} d^2p_1\, d^2p_2 \dots d^2p_N\, d^2q_1\, d^2q_2 \dots d^2q_N \\
&= \frac{1}{h^{2N}}\int_{E<\sum_i^{2N}\frac{p_i^2}{2m}+\frac{kq_i^2}{2}<E+\delta E} d^2p_1\, d^2p_2 \dots d^2p_N\, d^2q_1\, d^2q_2 \dots d^2q_N \\
&= \frac{1}{h^{2N}}\int_{E<\sum_i^{2N}(u_i^2+v_i^2)<E+\delta E} (2m)^N \left(\frac{2}{k}\right)^N du_1\, du_2 \dots du_{2N}\, dv_1\, dv_2 \dots dv_{2N} \\
&= \left(\frac{4m}{h^2k}\right)^N \frac{\partial \Lambda_{2N}}{\partial E}\delta E
\end{aligned}$$

$$\Lambda_{2N} = R^{2N}C_{4N} = E^{2N}\frac{\pi^{2N}}{(2N)!}$$

$$\frac{\partial \Lambda_{2N}}{\partial E}\delta E = 2NE^{2N-1}\frac{\pi^{2N}}{(2N)!}\delta E$$

$$\therefore \Omega_N = \left(\frac{4m}{h^2k}\right)^N \frac{E^{2N-1}\pi^{2N}}{(2N-1)!}\delta E = \left(\frac{4m\pi^2}{h^2k}\right)^N \frac{E^{2N-1}}{(2N-1)!}\delta E$$

$$\beta = \frac{1}{k_BT} = \frac{\partial}{\partial E}\ln\Omega_N = \frac{\partial}{\partial E}\left[(2N-1)\ln E\right] = \frac{2N-1}{E}$$

$$E = (2N-1)k_BT \approx 2Nk_BT \quad [\text{for } N \gg 1]$$

Hint: Use a change of variable, $u^2 = \frac{p^2}{2m}$ and $v^2 = \frac{kq^2}{2}$. Here k is the spring constant, and k_B is the Boltzmann constant. The expressions for Λ, R and C can be remembered as follows:

$$\Lambda_{\text{dimension}\times\text{no. of particles}} = R^{\text{dimension}\times\text{no. of particles}}C_{\text{no. of integrals}}$$

$$R = \text{the energy (in this case, } R = E)$$

$$C_n = \frac{\pi^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!}$$

Question 1 (ii)

$$Z_1 = \sum_i e^{-\beta\left(\frac{p_i^2}{2m}+\frac{kq_i^2}{2}\right)} = \frac{1}{h^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\beta\left(\frac{p^2}{2m}+\frac{kq^2}{2}\right)} d^2p\, d^2q$$

$$= \frac{1}{h^2}\left(\int_{-\infty}^{\infty} e^{-\frac{\beta p^2}{2m}}\,dp\right)^2 \left(\int_{-\infty}^{\infty} e^{-\frac{\beta k q^2}{2}}\,dq\right)^2$$

$$= \frac{1}{h^2}\left(\sqrt{\frac{2m\pi}{\beta}}\right)^2 \left(\sqrt{\frac{2\pi}{\beta k}}\right)^2 = \frac{m}{\hbar^2 k \beta^2} = \left(\frac{1}{\hbar\omega\beta}\right)^2$$

$$Z_N = \left(\frac{1}{\hbar\omega\beta}\right)^{2N}$$

$$E = -\frac{\partial}{\partial\beta}(\ln Z_N) = \frac{\partial}{\partial\beta}(2N\ln\beta) = 2Nk_B T$$

Question 1 (iii)

Characteristics of Microcanonical Ensemble:

- E, V, N constant.
- System isolated.
- Ω is hard to calculate.
- It has a specific total Energy E.

Characteristics of Canonical Ensemble:

- T, V, N constant.
- System not isolated, but in contact with a reservoir with temperature T.
- Energy can be exchanged with the heat bath.
- It assigns a probability to each distinct microstate through the partition function.
- There is a fluctuation in energy.

Hint: Well, we should as well add this in! For your reference.

Characteristics of Grand Canonical Ensemble:

- T, V, μ constant.
- It is in thermal and chemical equilibrium with a reservoir.
- It can exchange energy and particles with the reservoir.
- It assigns a probability to each distinct microstate through the grand partition function.
- The grand potential Ω_G is constant.
- Provides a natural setting for Fermi-Dirac & Bose-Einstein statistics.
- There is a fluctuation in energy and particle number.

Question 2 (i)

$$Z = \sum_{r}^{\infty} e^{-\beta\hbar\omega\left(r+\frac{1}{2}\right)} = e^{-\frac{\beta\hbar\omega}{2}} \sum_{r}^{\infty} e^{\beta\hbar\omega r} = \frac{e^{-\frac{x}{2}}}{1-e^{-x}} \quad \text{[shown]}$$

$$C = \frac{\partial U}{\partial T} = -\frac{\partial}{\partial T}\frac{\partial}{\partial \beta}(\ln Z) = \frac{\partial x}{\partial T}\frac{\partial}{\partial x}\left(\frac{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}\right)\frac{\hbar\omega}{2}$$

$$= \frac{x}{2T}\frac{4}{\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)^2}\frac{\hbar\omega}{2} = ke^{-x}\left(\frac{x}{1-e^{-x}}\right)^2 \quad \text{[shown]}$$

Question 2 (ii)

At high temperature, $x \ll 1$, so

$$C = ke^{-x}x^2(1-e^{-x})^{-2} \approx k(1-x)x^2(1-1+x)^{-2} = k(1-x) \approx k$$

Which is a constant. This is Curie's law.

At low temperature, $x \gg 1$, so

$$C = ke^{-x}x^2(1-e^{-x})^{-2} \approx ke^{-x}x^2 \approx 0$$

Hint: The graph of $y = \frac{e^{-\frac{1}{T}}}{T^2\left(1-e^{-\frac{1}{T}}\right)^2}$ looks like the one below. So indeed, C goes to zero as T tends to zero, because the exponential blows up before the quadratic term.

Question 2 (iii)

For N molecules, $Z_N = Z^N$, so

$$S = k\ln Z^N + \frac{U}{T} = Nk\ln\left(\frac{e^{-\frac{x}{2}}}{1-e^{-x}}\right) - \frac{1}{T}\frac{\partial}{\partial\beta}(\ln Z^N)$$

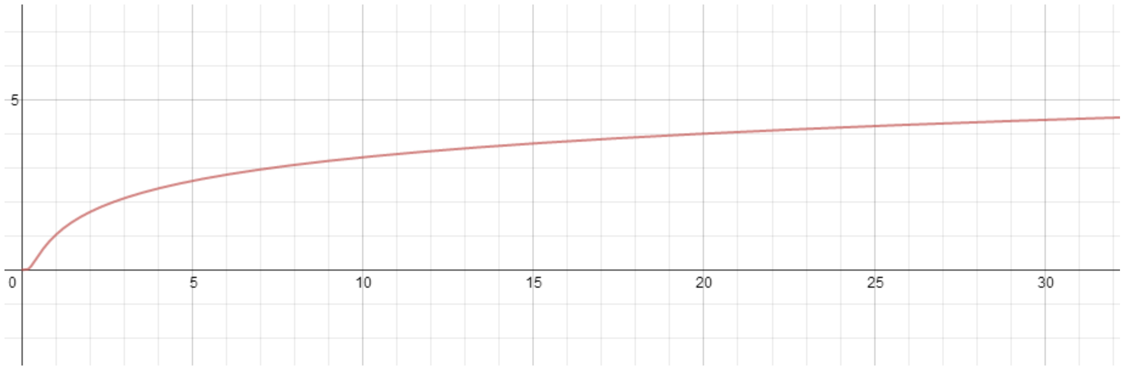
$$= Nk \ln\left(\frac{e^{-\frac{x}{2}}}{1-e^{-x}}\right) - \frac{1}{T}\frac{\partial x}{\partial \beta}\frac{\partial}{\partial x}\ln\left(\frac{e^{-\frac{x}{2}}}{1-e^{-x}}\right)$$

$$= Nk\left[\frac{xe^{-x}}{1-e^{-x}} - \ln(1-e^{-x})\right]$$

At low temperature,

$S \approx Nkxe^{-x} \approx 0$, and this agrees with the 3rd law of thermodynamics.

Hint: The graph of $y = \frac{e^{-\frac{1}{T}}}{T\left(1-e^{-\frac{1}{T}}\right)} - \ln\left(1-e^{-\frac{1}{T}}\right)$ is shown below. It increases indefinitely.



Question 3 (i)

In a 2D electron system, $p_z = 0$.

$$\mathcal{H} = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2$$

$$\mathcal{H}\psi = E\psi$$

$$\frac{1}{2m}\left(p^2 + \frac{2e}{c}\vec{p}\cdot\vec{A} + \frac{e^2}{c^2}A^2\right)\psi = E\psi$$

$$\frac{1}{2m}\left(p_x^2 + p_y^2 + \frac{2e}{c}p_y Hx + \frac{e^2}{c^2}H^2x^2\right)\psi = E\psi$$

$$\frac{1}{2m}\left[p_x^2 + \frac{e^2H^2}{c^2}\left(x + \frac{p_y c}{He}\right)^2\right]\psi = E\psi$$

$$\frac{1}{2m}[p_x^2 + m\omega_0^2(x - x_0)^2]\psi = E\psi$$

$$\omega_0 = \frac{eH}{mc}, \qquad x_0 = -\frac{p_y c}{eH}$$

So the eigenvalues are

$$E = \hbar\omega_0\left(n + \frac{1}{2}\right) = \frac{\hbar eH}{mc}\left(n + \frac{1}{2}\right), \qquad n = 0,1,2,\ldots$$

Question 3 (ii)

The Hall current,

$$j_y = -\sigma_{xy} E_x$$
$$nve = -\sigma_{xy} E_x$$
$$\sigma_{xy} = -\frac{nve}{E_x}$$

To balance a charge e moving with velocity in the y-direction, a magnetic force is required.

$$eE_x = -\frac{ev}{c} H$$
$$\Rightarrow \frac{1}{E_x} = -\frac{c}{vH}$$

$$\therefore \sigma_{xy} = -nve\left(-\frac{c}{vH}\right) = \frac{nec}{H} \quad \text{[shown]}$$

Question 3 (iii)

Quantum Hall effect happens under very strong magnetic field and low temperatures. Suppose there are just enough electrons to fill the lowest p Landau level, we have

$$\frac{nec}{H} = p\frac{e^2}{h} \quad \Rightarrow \quad n = p\frac{eH}{hc}$$

Question 3 (iv)

In a pure metal, a series of delta functions mark the positions of the Landau levels. In the presence of impurities, the Landau levels broaden to bands, and localized states fill up the gap between these bands. This is the Fermi level, it can lie in the continuum between Landau bands and it can shift in response to a change in occupancy of the filled Landau bands so that the band beneath it remains filled. Thus for a certain range of magnetic field, the lowest Landau bands remain completely filled, and hence the plateau.

Question 3 (v)

$\frac{h}{e^2} = 25813\Omega$ is the von Klitzing constant. It is defined to be so as a practical standard for the measurement of Ohm.

Question 4 (i)

$$E_{jz} = -g\mu_B J_z H$$

$$Z = \sum_{J_z}^{\infty} e^{-\beta E_{J_z}} = \sum_{J_z=-J}^{J} e^{\beta g \mu_B J_z H} = e^{-\beta g \mu_B J H} \sum_{J_z=0}^{2J} e^{\beta g \mu_B J_z H}$$
$$= e^{-\beta g \mu_B J H} \frac{1 - e^{\beta g \mu_B (2J+1) H}}{1 - e^{\beta g \mu_B H}} = \frac{\sinh y \left(J + \frac{1}{2}\right)}{\sinh \frac{y}{2}}$$

Where $y = g\beta\mu_B H$.

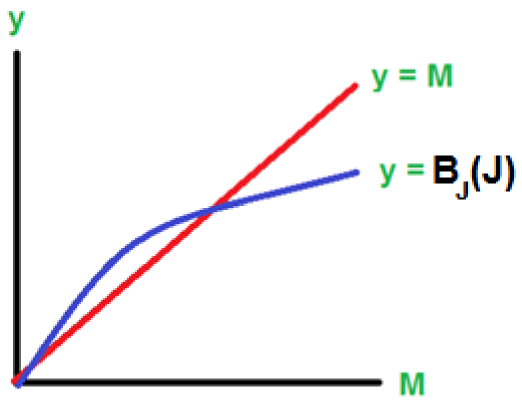
$$M = \frac{N}{\beta} \frac{\partial}{\partial H} (\ln Z) = NkT \frac{\partial}{\partial H} \left[\ln \sinh y \left(J + \frac{1}{2}\right) - \ln \sinh \frac{y}{2} \right]$$
$$= N\mu_B g \left[\left(J + \frac{1}{2}\right) \coth y \left(J + \frac{1}{2}\right) - \frac{1}{2} \coth \frac{y}{2} \right]$$
$$= N\mu_B g J B_J(y) \quad \text{[shown]}$$

Question 4 (ii)

$$H \Rightarrow H + H_m = H + \frac{2J_e n J_m}{g\mu_B}$$

$$\therefore M = N g \mu_B J_e m$$
$$= N\mu_B g J \left\{ \left(J + \frac{1}{2}\right) \coth \left[\beta g \mu_B \left(H + \frac{2J_e n J_m}{g\mu_B}\right) \left(J + \frac{1}{2}\right) \right] - \frac{1}{2} \coth \frac{1}{2} \beta g \mu_B \left(H + \frac{2J_e n J_m}{g\mu_B}\right) \right\}$$

So we have an equation $m = B_J(J)$.



One solution is $m = 0$, the other is $m \neq 0$. For $H = 0$, if close to $m = 0$, $\frac{\partial y}{\partial m} > 1$, then

$$B_J \approx \frac{y}{3}(J + 1) = \frac{2}{3} J(J + 1) \beta J_e n m$$

At this point $\beta = \frac{1}{kT_c}$, T_c is the Curie temperature.

$$\therefore T_c = \frac{2}{3k} J(J+1) J_e n$$

Solutions provided by:
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