# National University of Singapore

PC4241 Statistical Mechanics

(Semester I: AY2011-12, 29 November 2011)

Time Allowed: Two Hours

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**INSTRUCTIONS TO CANDIDATES**

1. This examination paper contains **FOUR** questions and comprises **THREE** printed pages.
2. All questions carry equal marks.
3. Answer any **THREE** questions.
4. This is a CLOSED BOOK examination.
5. One A4 double-sided sheet of formulae and equations is allowed.

1. Consider a system of $N$ 2-D harmonic oscillators, each with energy

$$\varepsilon = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}k\left(q_x^2 + q_y^2\right)$$

and classical angular frequency $\omega = \sqrt{k/m}$ .

(i) Use the microcanonical ensemble to calculate the number of states of the system with energy between $E$ and $E + \delta E$ . Hence obtain a relation between the total energy of the system and the temperature.

(ii) Use the canonical ensemble to determine the partition function of the system. Hence obtain the relation between the total energy and the temperature.

(iii) List the characteristics of microcanonical ensemble and canonical ensemble.

$$\left[ Volume\ of\ a\ unit\ sphere\ in\ \text{n-dimension}\ C_n = \frac{\pi^{\frac{n}{2}}}{\left(n/2\right)!}, \quad \int_{-\infty}^{\infty} exp(-x^2)dx = \pi^{\frac{1}{2}} \right]$$

2. According to quantum mechanics, a diatomic molecule possesses a set of nondegenerate vibrational energy levels with energy $\hbar\omega\left(r + \frac{1}{2}\right)$, $r = 0,1,2, \dots, \infty$.

(i) Show that the vibrational partition function $z$ and the heat capacity $c$ of the molecule are given respectively by $z = \frac{e^{\frac{-x}{2}}}{1-e^{-x}}$ , $c = ke^{-x}\left(\frac{x}{1-e^{-x}}\right)^2$ where $x = \beta\hbar\omega$.

(ii) Determine the high and low temperature limits of the heat capacity $c$ . Comment on the result.

(iii) Consider a crystal of $N$ diatomic molecules. Calculate its vibrational entropy, find its low temperature limit and comment on the result.

3. Consider a 2-D electron system in perpendicular magnetic field $H$.

(i) Write down the Schrödinger equation in the Landau gauge with $A = [0, Hx, 0]$. Hence, obtain the eigen values and degeneracy of the energy levels.

(ii) Show that in classical physics, the Hall conductivity is given by $\sigma_{xy} = nec/H$ , where $n$ is the number of electrons per unit area.

(iii) Show how the equation in (ii) explains the integer quantum Hall effect if the electrons just fill the first lowest $p$ Landau levels.

(iv) Experimentally, the Hall conductivity exhibits plateaus of value $pe^2/h$, $p =$ integer at Landau levels $p = \frac{nhc}{eH}$ . Explain the plateaus observed.

(v) Explain how the integer quantum Hall effect can be used to define the ohm.

4. (i) Consider a paramagnetic system of $N$ atoms in an external magnetic field $H$ in the $z$ direction. Energy of each atom is given by $E_{Jz} = -g\mu_B J_z H$ , $J_z = -J, -J+1, \dots, J$ . Assuming Maxwell-Boltzmann distribution, evaluate the partition function of the atom and show that the total magnetic moment of the system is given by

$M = Ng\mu_B J B_J(y)$ ,

where $y = \beta g \mu_B H$

and $B_J(y) = \frac{1}{J}\left[\left(J+\frac{1}{2}\right)\coth y\left(J+\frac{1}{2}\right) - \frac{1}{2}\coth\frac{y}{2}\right]$ .

(ii) In the mean-field approximation for ferromagnetic transition, $H$ is replaced by $H + H_m$ where $H_m$ is related to $M$ by $g\mu_B H_m = 2J_e nJm$ and $M = Ng\mu_B Jm$ , where $J_e$ is the exchange constant and $n$ the number of nearest neighbours. Using (i), find an equation determining $m$ , the order parameter. Hence obtain an expression for $T_c$ , the Curie temperature.

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