Question 1 (i)

We have N non-interacting and distinguishable particles. So

$$Z_1 = 1 + e^{-\beta E}$$

$$Z_N = \left(1 + e^{-\beta E}\right)^N, \qquad \ln Z_N = N \ln\left(1 + e^{-\beta E}\right)$$

$$U = -\frac{\partial}{\partial \beta}(\ln Z_N) = NE\frac{1}{e^{\beta E} + 1}$$

Question 1 (ii)

$$\frac{U}{NE} = \frac{1}{e^{\beta E} + 1}$$

$$e^{\beta E} = \frac{NE}{U} - 1$$

$$\frac{E}{kT} = \ln\left(\frac{NE}{U} - 1\right)$$

$$T = \frac{E}{k \ln\left(\frac{NE}{U} - 1\right)}$$

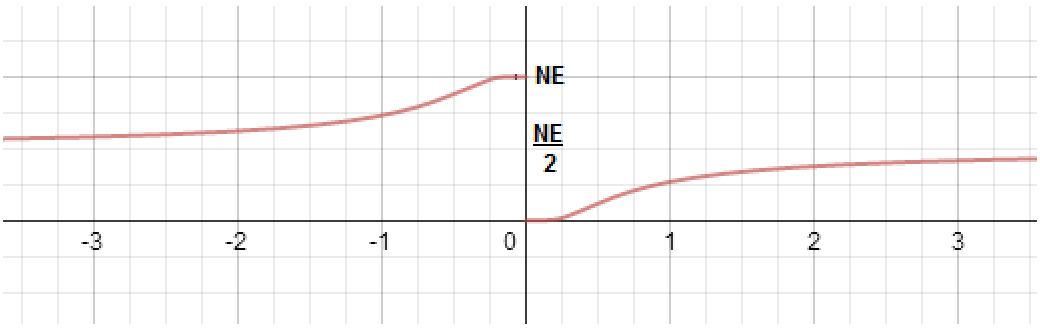
T is positive if $\frac{NE}{U} - 1 > 1$, which means $U < \frac{NE}{2}$.

T is negative when $0 < \frac{NE}{U} - 1 < 1$, which means $\frac{NE}{2} < U < NE$.

As $T \to 0^+$, $U = NEe^{-\beta E} \approx 0$; as $T \to 0^-$, $U = NE^{e^{\beta E}} \approx NE$.

As $T \to \infty$, $U = \frac{NE}{2}$.

So the graph looks like this:



Question 1 (ii)

$$C = \frac{\partial U}{\partial T} = \frac{\partial \beta}{\partial T}\frac{\partial U}{\partial \beta} = \frac{NE}{kT^2}\frac{Ee^{\beta E}}{(e^{\beta E} + 1)^2} = \frac{NE^2}{kT^2}\frac{e^{\beta E}}{(e^{\beta E} + 1\,)^2}$$

At high T, β is small, so

$$C \approx \frac{NE^2}{kT^2}\frac{1+\beta E}{(1+\beta E+1)^2} \approx \frac{NE^2}{kT^2}\frac{1}{4}(1+\beta E)\left(1-\frac{\beta E}{2}\right) \approx \frac{1}{4}\frac{NE^2}{kT^2} \approx 0$$

At low T, β is big, so

$$C = \frac{NE^2}{kT^2}e^{-\beta E} \approx 0$$

Because the exponent blows up before the $\frac{1}{T^2}$ term.

Hint: In actual fact, the graph looks something like this:

Question 1 (iv)

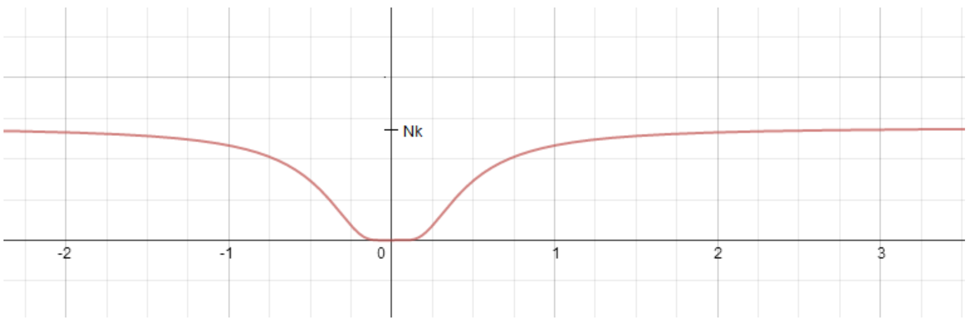
$$S = \frac{U}{T} + k\ln Z_N = \frac{NE}{T}\frac{1}{e^{\beta E}+1} + Nk\ln\left(1+e^{-\beta E}\right) = Nk\left[\frac{\beta E}{e^{\beta E}+1} + \ln\left(1+e^{-\beta E}\right)\right]$$

At high temperature,

$$S \approx Nk\left[\frac{\beta E}{2} + 1 - \beta E\right] = Nk\left(1-\frac{E}{2kT}\right) \approx Nk$$

At low temperature,

$$S \approx Nk\beta E e^{-\beta E} = \frac{NE}{T}e^{-\frac{E}{kT}} \approx 0$$



Question 2 (i)

$$Z = V\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}}$$

$$\mathcal{Z} = \sum_N e^{-\beta(E-N\mu)} = \sum_N \frac{Z^N}{N!}\zeta^N = e^{Z\zeta} = \exp\left[V\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} e^{\mu\beta}\right] \quad \text{[shown]}$$

Hint: make use of the power series of the exponent.

Question 2 (ii)

$$\Omega_G = -kT\ln\mathcal{Z} = -kTV\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} e^{\mu\beta} = -PV$$

$$P = kT\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} e^{\mu\beta} = \frac{\partial\Omega_G}{\partial V}$$

$$N = \frac{1}{\beta}\frac{\partial}{\partial\mu}(\ln\mathcal{Z}) = \frac{\partial}{\partial\mu}(kT\ln\mathcal{Z}) = -\left(\frac{\partial\Omega_G}{\partial\mu}\right)_{\beta,V} = V\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} e^{\mu\beta}$$

$$\therefore P = kT\left(\frac{N}{V}\right) \quad\Rightarrow\quad PV = NkT$$

Question 2 (iii)

$$(\Delta N)^2 = \langle N^2\rangle - \langle N\rangle^2$$

$$\langle N^2\rangle = \frac{\sum_s N^2 e^{-\beta E_s + N_s\mu\beta}}{\mathcal{Z}} = \frac{1}{\mathcal{Z}}\frac{\partial^2}{\partial\mu^2}\left(\sum_s e^{-\beta E_s + N_s\mu\beta}\right) = \frac{1}{\mathcal{Z}\beta^2}\frac{\partial^2\mathcal{Z}}{\partial\mu^2}$$

$$= \frac{1}{\mathcal{Z}}\frac{1}{\beta^2}\frac{\partial}{\partial\mu}(\langle N\rangle\beta\mathcal{Z}) = \langle N\rangle\frac{1}{\beta\mathcal{Z}}\frac{\partial\mathcal{Z}}{\partial\mu} + \frac{1}{\beta}\left(\frac{\partial\langle N\rangle}{\partial\mu}\right)_{\beta,x} = \langle N\rangle^2 + kT\left(\frac{\partial\langle N\rangle}{\partial\mu}\right)_{\beta,x}$$

$$\therefore (\Delta N)^2 = kT\left(\frac{\partial\langle N\rangle}{\partial\mu}\right)_{\beta,x}$$

Question 2 (iv)

$$\langle N\rangle = V\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} e^{\mu\beta}$$

$$(\Delta N)^2 = kT\left(\frac{\partial\langle N\rangle}{\partial\mu}\right)_{\beta,x} = V\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} e^{\mu\beta} = \langle N\rangle$$

$$\frac{\Delta N}{\langle N\rangle} = \frac{\sqrt{\langle N\rangle}}{\langle N\rangle} = \frac{1}{\sqrt{\langle N\rangle}}$$

This follows the Poisson distribution. So the fluctuation in particle number is negligible if the system is large.

Question 3 (i)

$$\frac{\lambda^3 P}{kT} = f_{\frac{5}{2}}(\zeta), \qquad \frac{\lambda^3 N}{V} = f_{\frac{3}{2}}(\zeta)$$

$$PV = kT\ln\mathcal{Z} \quad\Rightarrow\quad \ln\mathcal{Z} = \frac{V}{\lambda^3} f_{\frac{5}{2}}(\zeta)$$

$$U = -\frac{\partial}{\partial\beta}\left[\frac{V}{\lambda^3} f_{\frac{5}{2}}(\zeta)\right] = \frac{3}{2\beta}\frac{V}{\lambda^3} f_{\frac{5}{2}}(\zeta) = \frac{3}{2}PV \quad \text{[shown]}$$

Question 3 (ii)

$$f_{\frac{3}{2}}(\zeta) = \frac{\lambda^3 N}{V} = \zeta - \frac{\zeta^2}{2^{\frac{3}{2}}} - \cdots$$

1st order,

$$\frac{\lambda^3 N}{V} = \zeta$$

2nd order,

$$\frac{\lambda^3 N}{V} = \zeta - \frac{1}{2^{\frac{3}{2}}}\left(\frac{\lambda^3 N}{V}\right)^2$$

$$e^{\mu\beta} = \frac{\lambda^3 N}{V}\left[1 + \frac{1}{2^{\frac{3}{2}}}\left(\frac{\lambda^3 N}{V}\right)\right]$$

$$\mu\beta = \ln\left(\frac{N\lambda^3}{V}\right) + \ln\left[1 + \frac{1}{2^{\frac{3}{2}}}\left(\frac{\lambda^3 N}{V}\right)\right]$$

$$\mu \approx kT\left[\ln\left(\frac{N\lambda^3}{V}\right) + \frac{1}{2^{\frac{3}{2}}}\left(\frac{\lambda^3 N}{V}\right)\right] \quad \text{[shown]}$$

Question 3 (iii)

$$\frac{N\lambda^3}{V} \approx \frac{4}{3\sqrt{\pi}}\left[(\ln\zeta)^{\frac{3}{2}} + \frac{\pi^2}{8}\ (\ln\zeta)^{-\frac{1}{2}}\right]$$

1st order,

$$\left(\frac{3\sqrt{\pi}N\lambda^3}{4V}\right)^{\frac{2}{3}} = \ln\zeta = \frac{\mu}{kT} \approx \frac{\varepsilon_F}{kT} = \frac{T_F}{T}$$

2nd order,

$$\frac{N\lambda^3}{V} \approx \frac{4}{3\sqrt{\pi}}\left[(\ln\zeta)^{\frac{3}{2}} + \frac{\pi^2}{8}\left(\frac{4V}{3\sqrt{\pi}N\lambda^3}\right)^{\frac{1}{2}}\right]$$

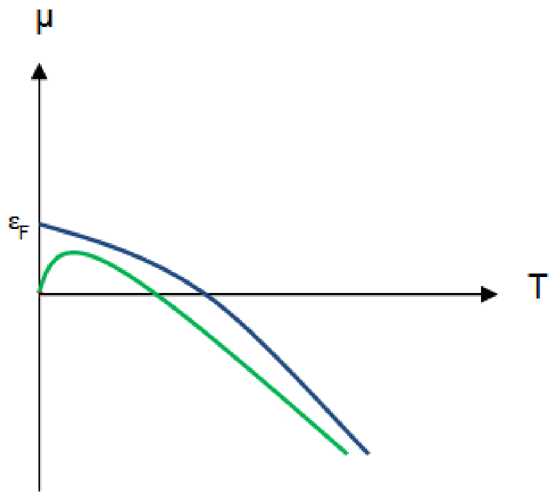
$$\left(\frac{T_F}{T}\right)^{\frac{3}{2}} = (\ln\zeta)^{\frac{3}{2}} + \frac{\pi^2}{8}\left(\frac{T}{T_F}\right)^{\frac{1}{2}}$$

$$(\ln\zeta)^{\frac{3}{2}} = \left(\frac{T_F}{T}\right)^{\frac{3}{2}}\left[1 - \frac{\pi^2}{8}\left(\frac{T_F}{T}\right)^2\right]$$

$$\ln\zeta = \frac{T_F}{T}\left[1 - \frac{\pi^2}{8}\left(\frac{T_F}{T}\right)^2\right]^{\frac{2}{3}} \approx \frac{T_F}{T}\left[1 - \frac{\pi^2}{12}\left(\frac{T_F}{T}\right)^2\right]$$

$$\therefore \mu = kT_F\left[1 - \frac{\pi^2}{12}\left(\frac{T_F}{T}\right)^2\right] \quad \text{[shown]}$$

Question 3 (iv)



The blue line is for fermions and the green line is for classical particles. The line for fermions does not start from $\mu = 0$, and it is always higher than the classical results.

Question 4 (i)

$$\mathcal{H} = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 = \frac{1}{2m}\left(p^2 + \frac{2e}{c}\vec{p}\cdot\vec{A} + \frac{e^2}{c^2}A^2\right)$$

$$\mathcal{H}\psi = E\psi$$

$$\frac{1}{2m}\left(p^2 + \frac{2e}{c}\vec{p}\cdot\vec{A} + \frac{e^2}{c^2}A^2\right)\psi = E\psi$$

$$\frac{1}{2m}\left(p_x^2 + p_y^2 + \frac{2e}{c}p_y Hx + \frac{e^2}{c^2}H^2x^2 + p_z^2\right)\psi = E\psi$$

$$\frac{1}{2m}\left[p_x^2 + \frac{e^2H^2}{c^2}\left(x + \frac{p_y c}{He}\right)^2 + p_z^2\right]\psi = E\psi$$

$$\left[\frac{p_x^2}{2m} + \frac{e^2H^2}{2mc^2}\left(x + \frac{p_y c}{eH}\right)^2\right]\psi = \left(E - \frac{p_z^2}{2m}\right)\psi$$

It is like a simple harmonic oscillator, so we have

$$E - \frac{p_Z^2}{2m} = \hbar\frac{eH}{mc}\left(j + \frac{1}{2}\right)$$

$$E = \frac{p_Z^2}{2m} + \hbar\omega_0\left(j + \frac{1}{2}\right) \quad \text{[shown]}$$

The energy is independent of k_y. For each j, there's a degeneracy equal to the allowed values of k_y. Consider $k_y = \frac{p_y}{\hbar} = \frac{2\pi n_y}{L}$. $x_0 = \frac{\hbar k_y c}{eH}$ must lie inside the volume L^3, so

$$0 < x_0 < L$$

$$0 < \frac{\hbar c}{eH}k_y < L$$

$$0 < \frac{\hbar c}{eH}\frac{2\pi n_y}{L} < L$$

$$0 < \frac{hcn_y}{eH} < L^2$$

$\therefore$ the degeneracy, $g = \frac{eH}{hc}L^2$

Question 4 (ii)

$$\mathcal{Z} = \prod_{\gamma}\left(1 + \zeta e^{\beta E_\lambda}\right)$$

$$\ln\mathcal{Z} = \sum_{\alpha=1}^{g}\sum_{j=0}^{\infty}\sum_{p_z}\ln\left(1 + \zeta e^{-\beta E}\right)$$

$$= 2g\sum_{j=0}^{\infty}\frac{L}{h}\int_0^{\infty} dp_z \ln\left[1+\zeta e^{-\beta E(p_z,j)}\right]$$

$$= \frac{2gL}{h}\sum_{j=0}^{\infty}\int_0^{\infty} dp_z \ln\left[1+\zeta e^{-\beta E(p_z,j)}\right] \quad \text{[shown]}$$

Hint: in this question, $\gamma = (p_z, j, \alpha)$ and $\alpha = 1,2,\ldots,g$.

Question 4 (iii)

At high temperature, β is small,

$$\ln \mathcal{Z} = \frac{2gL}{h}\sum_{j=0}^{\infty}\int_0^{\infty} dp_z\, \zeta e^{-\beta E(p_z,j)} = \frac{2gL}{h}\sum_{j=0}^{\infty}\int_0^{\infty} dp_z\, e^{-\beta\left[N\mu-\frac{p_z^2}{2m}-\hbar\omega_0\left(j+\frac{1}{2}\right)\right]}$$

$$= \frac{2gL}{h}\zeta\int_0^{\infty} e^{-\beta\frac{p_z^2}{2m}}\, dp_z \sum_{j=0}^{\infty} e^{\beta\hbar\omega_0\left(j+\frac{1}{2}\right)}$$

$$= \frac{2gL}{2h}\zeta\sqrt{\frac{2\pi m}{\beta}}\, e^{-\frac{\beta\hbar\omega_0}{2}}\frac{1}{1-e^{-\beta\hbar\omega_0}}$$

$$= \frac{\zeta g L}{\lambda}\frac{e^{-x}}{1-e^{-2x}} \quad \text{[shown]}$$

$$\ln \mathcal{Z} = \frac{gL\zeta}{\lambda}\frac{1}{e^{x}-e^{-x}}$$

$$\approx \frac{gL\zeta}{\lambda}\frac{1}{\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}-1+x-\frac{x^2}{2}+\frac{x^3}{6}\right)}$$

$$= \frac{gL\zeta}{\lambda}\frac{1}{\beta\hbar\omega_0}\left[1+\frac{1}{6}\left(\frac{\beta\hbar\omega_0}{2}\right)^2\right]^{-1} \approx \frac{gL\zeta}{\lambda}\frac{1}{\beta\hbar\omega_0}\left[1-\frac{(\beta\hbar\omega_0)^2}{24}\right]$$

$$= \frac{2\pi L^3\zeta m}{h^2\beta\lambda}\left[1-\frac{1}{24}\left(\frac{\hbar\omega_0}{kT}\right)^2\right] = \frac{\zeta L^3}{\lambda^3}\left[1-\frac{1}{24}\left(\frac{\hbar\omega_0}{kT}\right)^2\right]$$

$$= N\left[1-\frac{1}{24}\left(\frac{\hbar\omega_0}{kT}\right)^2\right] \quad \text{[shown]}$$

Hint: use the formula of sum to infinity for the geometric series. Also, remember that $\frac{N}{L^3} = \frac{\zeta}{\lambda^3}$.

Question 4 (iv)

$$M = \frac{1}{\beta}\frac{\partial}{\partial H}(\ln \mathcal{Z}) = kT\,\frac{\partial\omega_0}{\partial H}\frac{\partial}{\partial\omega_0}\left\{N\left[1-\frac{1}{24}\left(\frac{\hbar\omega_0}{kT}\right)^2\right]\right\}$$

$$= -\frac{N}{kT}\frac{1}{12}\frac{e^2\hbar^2}{m^2c^2}H = -\frac{N}{3kT}\mu_0^2 H \quad \text{[shown]}$$

We see that $M \propto \frac{1}{T}$, this is in accordance to Curie's Law.

Solutions provided by:

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