National University of Singapore

PC4241 Statistical Mechanics

(Semester I: AY2012-13, 3 December 2012)

Time Allowed: Two Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR** questions and comprises **FOUR** printed pages.
2. All questions carry equal marks.
3. Answer any **THREE** questions.
4. This is a CLOSED BOOK examination.
5. One A4 double-sided sheet of formulae and equations is allowed.

1. A system of N non-interacting distinguishable particles, each of which has only two non-degenerate energy levels 0 and $E(>0)$, is in thermal equilibrium at temperature T.

(i) Calculate the partition function Z_N and the energy U of the system.

(ii) Obtain an expression for the temperature T and show that T can be positive or negative depending on the value of U. Calculate the limiting values of U near $T = 0$ and $T \to \pm\infty$. Make a sketch of U vs T.

(iii) Calculate the heat capacity C of the system and its high- and low- temperature limits. Make a sketch of C vs T.

(iv) Calculate the entropy S of the system and its high- and low- temperature limits. Make a sketch of S vs T.

2. The single molecule partition function z for an ideal gas consisting of N identical monatomic molecules, each of mass m, enclosed in a compartment V at temperature T is given by

$$z = V\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} \quad , \text{ where } \beta = \frac{1}{kT} \quad .$$

(i) Show that the grand partition function $\mathcal{Z}$ is given by

$$\mathcal{Z} = exp\left[e^{\beta\mu}V\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}}\right] \quad , \text{ where } \mu \text{ is the chemical potential.}$$

(ii) Obtain an expression for the grand potential Ω_G of the gas and show that its pressure P and average number of molecules $\langle N\rangle$ are given respectively by

$$P = -\left(\frac{\partial\Omega_G}{\partial V}\right)_{\beta,\beta\mu} \quad , \quad \langle N\rangle = -\left(\frac{\partial\Omega_G}{\partial\mu}\right)_{\beta,V} \quad .$$

Hence, obtain the equation of state for the gas.

(iii) Show that the number fluctuation ΔN, where $(\Delta N)^2 = \langle N^2\rangle - \langle N\rangle^2$, is given by $(\Delta N)^2 = \frac{1}{\beta}\left(\frac{\partial\langle N\rangle}{\partial\mu}\right)_{\beta,V}$.

(iv) Calculate $(\Delta N)^2$ and find the relative fluctuation $\Delta N/\langle N\rangle$ of the number of molecules. Comment on the result.

3. The parametric equations of state for a Fermi gas are given by

$\frac{\lambda^3 P}{kT} = f_{\frac{5}{2}}(z)$, $\frac{\lambda^3 N}{V} = f_{\frac{3}{2}}(z)$, where λ is the thermal wavelength, z the fugacity and $f_k(z)$ the Fermi function.

(i) Obtain the energy U of the gas as a function of $f_{\frac{5}{2}}(z)$ and hence show that $U = \frac{3}{2}PV$.

(ii) At high temperatures, $f_{\frac{3}{2}}(z) \approx z - \frac{z^2}{2^{3/2}}$. Use this result to show that the chemical potential μ is given by

$$\mu \approx kT\left[ln\left(\frac{N\lambda^3}{V}\right) + \frac{N}{V}\frac{\lambda^3}{2^{3/2}}\right] .$$

(iii) At low temperatures,

$$f_{\frac{3}{2}}(z) \approx \frac{4}{3\sqrt{\pi}}\left[(lnz)^{\frac{3}{2}} + \frac{\pi^2}{8}\frac{1}{\sqrt{lnz}}\right] .$$

Use this result to show that

$$\mu \approx \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right] ,$$

where ϵ_F is the Fermi energy and T_F the Fermi temperature.

(iv) Sketch the graph of μ vs T for the Fermi gas. Compare the result with that for the classical gas.

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4. A system of N free electrons in a uniform magnetic field H is confined in a cubic box of volume L^3.

(i) Ignoring the spin of the electron, write down the Schrödinger equation in the Landau gauge with $A = [0, Hx, 0]$. Hence, show that the eigenvalue $E(p_z, j)$ and degeneracy g of the energy levels are given respectively by

$E(p_{z,J}) = \frac{p_z^2}{2m} + \hbar\omega_0\left(j + \frac{1}{2}\right)$, $j = 0,1, \ldots$, where $\omega_0 = \frac{eH}{mc}$, and

$g = \left(\frac{eH}{hc}\right) L^2$.

(ii) Show that the grand partition function $\mathcal{Z}$ is given by

$ln\mathcal{Z} = \frac{2gL}{h}\sum_{j=0}^{\infty}\int_0^{\infty} dp_z ln\left[1 + ze^{-\beta E(p_{z,},j)}\right]$, where z is the fugacity.

(iii) In the high-temperature limit $z \ll 1$, $ln\mathcal{Z}=\frac{2zgL}{h}\sum_{j=0}^{\infty}\int_0^{\infty} dp_z e^{-\beta E(p_{z,},j)}$.

Show that $ln\mathcal{Z} = \frac{zgL}{\lambda}\frac{e^{-x}}{1-e^{-2x}}$, where $\lambda = \left(\frac{h^2\beta}{2\pi m}\right)^{\frac{1}{2}}$, $x = \frac{\beta\hbar\omega_0}{2}$.

Hence show that $ln\mathcal{Z} = N\left[1 - \frac{1}{24}\left(\frac{\hbar\omega_0}{kT}\right)^2\right]$.

(iv) Show that the magnetization M at high temperatures is given by $M = -\frac{N}{3kT}\mu_o^2 H$, where $\mu_o = \left(\frac{e\hbar}{2mc}\right)$. Comment on the result.

Hints: $\int_0^{\infty} e^{-q^2x^2}dx=\frac{\sqrt{\pi}}{2q}$, $\frac{e^{-x}}{1-e^{-2x}} \approx \frac{1}{2x}\left(1-\frac{x^2}{6}\right)$ for $x \ll 1$.

Ng S C

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