NATIONAL UNIVERSITY OF SINGAPORE

PC4243: Atomic & Molecular Physics II

(Semester 2: AY 2012-13)

Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This exam paper contains **FOUR** questions and comprises **SIX** printed pages.
2. You have to answer **THREE** out of the four questions.
3. Non-programmable calculators are permitted.
4. This is a **CLOSED BOOK** examination but **ONE** A4 sheet of hand written notes is permitted.
5. Please use only the supplied answer books, and don't mix answers to different problems on the same sheet.
6. There is a table of Clebsch-Gordan coefficients attached.

1: Atomic Structure

(a) Explain what is meant by the central field approximation and how it leads to the concept of electron configurations. Explain how perturbations arising from the residual electrostatic interactions and the spin-orbit interaction modify the structure of an isolated multi-electron configuration, in the LS-coupling limit.

(b) Neutral Strontium has a ground state configuration [Kr] $5s^2$. The excited state electron configurations 5s5p, and 5s6s have energy levels, measured from the ground state, of 14318, 14504, 14899, 21698, 29039, and 30592 (in units of cm^{-1}). The spectrum is found to have lines at 460.8 nm, 679.3 nm, 688.0 nm, 689.5 nm and 707.2 nm with Einstein A coefficients $28.17, 9.18, 24.65, 0.068$, and 29.37 respectively (in units of $10^6\,\mathrm{s}^{-1}$).

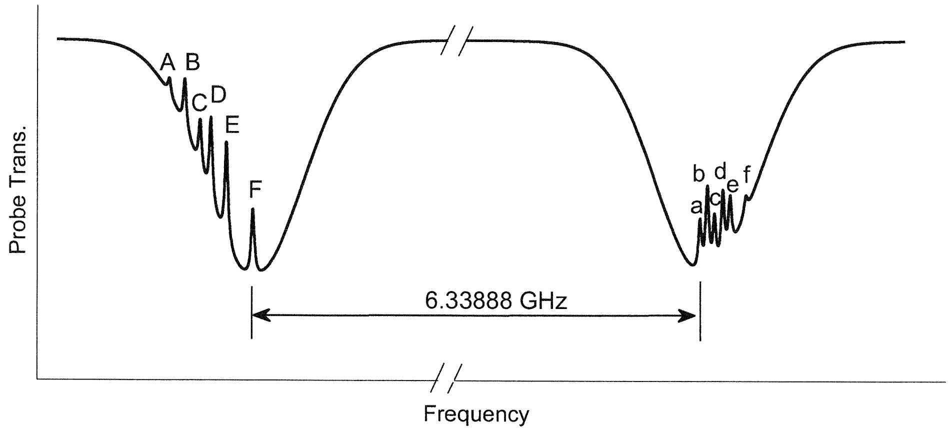
   (i) Suggest, with reasons, appropriate quantum numbers to identify the given levels.

   (ii) Draw an energy level diagram including the given energy levels and transitions.

   (iii) Explain why the transition at 689.5 nm has a small decay rate relative to the other lines.

   (iv) Suggest two possible transitions from the ground state that might be suitable for making a magneto-optic trap. Which transition would yield the largest number of atoms? Which transition would yield the coldest atoms? Explain your reasoning.

2: Absorption Spectroscopy



A	B	C	D	E	F
0	78.47	156.94	211.8	290.27	423.60
a	b	c	d	e	f
0	36.11	72.22	114.59	150.70	229.17

The figure shows the Doppler free saturated absorption spectrum obtained from the 5 $^2\mathrm{S}_{1/2} \longleftrightarrow 5\ ^2\mathrm{P}_{3/2}$ transition in a vapour of atomic Rubidium (^{87}Rb). The relative positions, in MHz, of the saturated absorption peaks in each group are given in the table.

(a) *Briefly* explain the principle of Doppler-free saturation spectroscopy. Explain the appearance of cross-over resonances in the spectrum of multi-level atoms.

(b) Show that the hyperfine interaction $H_{\mathrm{hfs}} = A_{nlj}\mathbf{I}\cdot\mathbf{J}$ leads to an interval rule which may be written

$$\Delta E_{F,F-1} = \Delta E_F - \Delta E_{F-1} = A_{nlj}F.$$

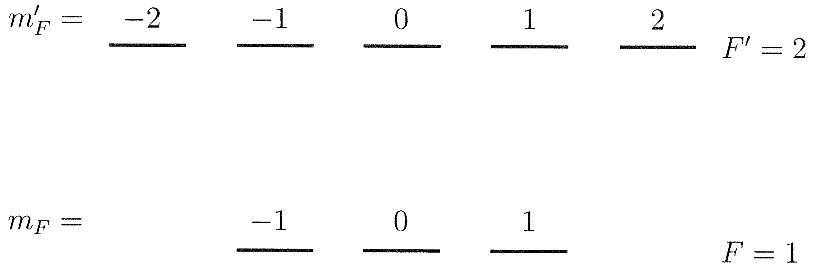
(c) Determine the hyperfine splittings for the 5 $^2\mathrm{S}_{1/2}$ and 5 $^2\mathrm{P}_{3/2}$ levels.

(d) Determine the extent to which the interval is obeyed for the Rubidium spectrum shown above and estimate the nuclear spin, I.

— Please turn over —

3: Laser Cooling

Consider an atom that has a lower level with $F = 1$ and an upper level with $F' = 2$ as depicted below.



(a) Determine the transition rates for all electric dipole-transitions between the two levels in terms of the total upper state decay rate, Γ.

(b) Suppose that the atom moves through an electric field given by

$$\mathbf{E} = |\mathbf{E}_0| \operatorname{Re}\left[e^{-i\omega t}\left(\cos(kz)\hat{\mathbf{e}}_+ + i\sin(kz)\hat{\mathbf{e}}_-\right)\right]$$

where $\hat{\mathbf{e}}_\pm$ are the unit vectors for $\sigma^\pm$ polarization. Determine the light shift potentials as a function of position for the $m_F = 0, \pm 1$ ground-states in terms of the laser detuning Δ, electric field $E_0 = |\mathbf{E}_0|$, and dipole matrix element

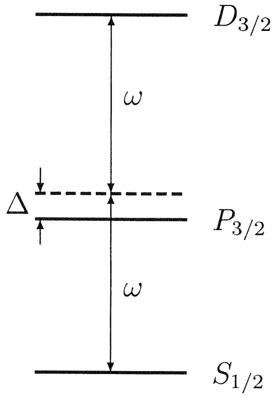
$$\mu = \langle\, F' = 2, m'_F = 2 \,|\, e\mathbf{r} \cdot \hat{\mathbf{e}}_+ |\, F = 1, m_F = 1 \,\rangle .$$

(c) Explain the principle of Sisyphus cooling. How does the final temperature scale with the intensity, I, and detuning Δ of the laser beams? What determines the ultimate limit to the cooling process?

(d) Explain how a magnetic field might influence the sub-Doppler cooling process in this system.

— Please turn over —

4: Two photon transitions and AC Stark shifts



The figure shows the energy level diagram for the $5^2\mathrm{S}_{1/2}$, $5^2\mathrm{P}_{3/2}$, and $5^2\mathrm{D}_{3/2}$ levels of ^{87}Rb which has a nuclear spin of $I = 3/2$. The $5^2\mathrm{P}_{3/2}$ and $5^2\mathrm{D}_{3/2}$ levels are at $12816.545\,\mathrm{cm}^{-1}$ and $25700.536\,\mathrm{cm}^{-1}$ respectively from the $5^2\mathrm{S}_{1/2}$ ground state. The decay rates from $5^2\mathrm{P}_{3/2}$ to $5^2\mathrm{S}_{1/2}$ and from $5^2\mathrm{D}_{3/2}$ to $5^2\mathrm{P}_{3/2}$ are $2\pi \times 6.025\,\mathrm{MHz}$ and $2\pi \times 388.0\,\mathrm{kHz}$ respectively. The $5^2\mathrm{S}_{1/2}$ to $5^2\mathrm{D}_{3/2}$ two photon transition is driven by a π polarized laser field which resonantly couples the $|\,F = 2, m_F = 2\,\rangle$ hyperfine ground state to the $|\,F = 3, m_F = 2\,\rangle$ hyperfine upper state. The intensity of the laser field is $1\,\mathrm{MWm}^{-2}$.

(a) Determine the expansion of the $|\,F = 2, m_F = 2\,\rangle$ ground and $|\,F = 3, m_F = 2\,\rangle$ upper states in terms of the $|\,I, m_I\,\rangle|\,J, m_J\,\rangle$ bases.

(b) Calculate the Rabi rate for the two photon transition, assuming that the detuning, Δ, is much larger than the $5^2\mathrm{P}_{3/2}$ hyperfine structure.

(c) Calculate the induced AC Stark shifts of the ground and upper state levels due to the driving field.

(d) If the transition was to be driven in a room temperature vapour cell, explain how Doppler broadening could be eliminated. Suggest other possible mechanisms that might influence the transition frequency and linewidth.

Note: You may find the following equations useful

$$I_0 = \frac{1}{2}\epsilon_0 c E_0^2, \qquad A_{ij} = \frac{\omega_{ij}^3\,\mu_{ij}^2}{3\pi\epsilon_0\hbar c^3}$$

$$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{F/m}, \quad c = 2.9979 \times 10^8\,\mathrm{m/s}, \quad \hbar = 1.055 \times 10^{-34}\,\mathrm{Js}$$

[MDB]

— End of paper —

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

	J	J …
	M	M …
m_1	m_2	
m_1	m_2	Coefficients
.	.	
.	.	

1/2×1/2

m_1 m_2	1
	+1
+1/2 +1/2	1

m_1 m_2	1	0
	0	0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	1
	−1
−1/2 −1/2	1

$$Y_1^0=\sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1=-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0=\sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta-\frac{1}{2}\Big)$$

$$Y_2^1=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2=\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

2×1/2

m_1 m_2	5/2
	+5/2
+2 +1/2	1

m_1 m_2	5/2	3/2
	+3/2	+3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	5/2	3/2
	+1/2	+1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	5/2	3/2
	−1/2	−1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	5/2	3/2
	−3/2	−3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	5/2
	−5/2
−2 −1/2	1

1×1/2

m_1 m_2	3/2
	+3/2
+1 +1/2	1

m_1 m_2	3/2	1/2
	+1/2	+1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	3/2	1/2
	−1/2	−1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	3/2
	−3/2
−1 −1/2	1

3/2×1/2

m_1 m_2	2
	+2
+3/2 +1/2	1

m_1 m_2	2	1
	+1	+1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	2	1
	0	0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	2	1
	−1	−1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	2
	−2
−3/2 −1/2	1

2×1

m_1 m_2	3
	+3
+2 +1	1

m_1 m_2	3	2
	+2	+2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	3	2	1
	+1	+1	+1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

m_1 m_2	3	2	1
	0	0	0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	3	2	1
	−1	−1	−1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	3	2
	−2	−2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	3
	−3
−2 −1	1

3/2×1

m_1 m_2	5/2
	+5/2
+3/2 +1	1

m_1 m_2	5/2	3/2
	+3/2	+3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	5/2	3/2	1/2
	+1/2	+1/2	+1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	5/2	3/2	1/2
	−1/2	−1/2	−1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	5/2	3/2
	−3/2	−3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	5/2
	−5/2
−3/2 −1	1

1×1

m_1 m_2	2
	+2
+1 +1	1

m_1 m_2	2	1
	+1	+1
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	2	1	0
	0	0	0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	2	1
	−1	−1
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	2
	−2
−1 −1	1

$$Y_\ell^{-m}=(-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0}=\sqrt{\frac{4\pi}{2\ell+1}}\,Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2|j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1|j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m}=(-1)^{m-m'}d^{\,j}_{m,m'}=d^{\,j}_{-m,-m'}$$

3/2×3/2

m_1 m_2	3
	+3
+3/2 +3/2	1

m_1 m_2	3	2
	+2	+2
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m_1 m_2	3	2	1
	+1	+1	+1
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m_1 m_2	3	2	1	0
	0	0	0	0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m_1 m_2	3	2	1
	−1	−1	−1
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m_1 m_2	3	2
	−2	−2
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m_1 m_2	3
	−3
−3/2 −3/2	1

$$d^{\,1}_{0,0}=\cos\theta \qquad d^{\,1/2}_{1/2,1/2}=\cos\frac{\theta}{2} \qquad d^{\,1}_{1,1}=\frac{1+\cos\theta}{2}$$

$$d^{\,1/2}_{1/2,-1/2}=-\sin\frac{\theta}{2} \qquad d^{\,1}_{1,0}=-\frac{\sin\theta}{\sqrt{2}}$$

$$d^{\,1}_{1,-1}=\frac{1-\cos\theta}{2}$$

2×3/2

m_1 m_2	7/2
	+7/2
+2 +3/2	1

m_1 m_2	7/2	5/2
	+5/2	+5/2
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m_1 m_2	7/2	5/2	3/2
	+3/2	+3/2	+3/2
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

m_1 m_2	7/2	5/2	3/2	1/2
	+1/2	+1/2	+1/2	+1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

m_1 m_2	7/2	5/2	3/2	1/2
	−1/2	−1/2	−1/2	−1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

m_1 m_2	7/2	5/2	3/2
	−3/2	−3/2	−3/2
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

m_1 m_2	7/2	5/2
	−5/2	−5/2
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m_1 m_2	7/2
	−7/2
−2 −3/2	1

2×2

m_1 m_2	4
	+4
+2 +2	1

m_1 m_2	4	3
	+3	+3
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m_1 m_2	4	3	2
	+2	+2	+2
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

m_1 m_2	4	3	2	1
	+1	+1	+1	+1
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

m_1 m_2	4	3	2	1	0
	0	0	0	0	0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

m_1 m_2	4	3	2	1
	−1	−1	−1	−1
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

m_1 m_2	4	3	2
	−2	−2	−2
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m_1 m_2	4	3
	−3	−3
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m_1 m_2	4
	−4
−2 −2	1

$$d^{\,3/2}_{3/2,3/2}=\frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2}=-\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2}=\sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2}=-\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2}=\frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2}=-\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2}=\Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1}=-\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0}=\frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1}=-\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2}=\Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1}=\frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0}=-\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1}=\frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0}=\Big(\frac{3}{2}\cos^2\theta-\frac{1}{2}\Big)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).