NATIONAL UNIVERSITY OF SINGAPORE

PC4243: Atomic & Molecular Physics II

(Semester 2: AY 2013-14)

Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This exam paper contains **FOUR** questions and comprises **SIX** printed pages.
2. You have to answer **THREE** out of the four questions.
3. Non-programmable calculators are permitted.
4. This is a **CLOSED BOOK** examination but **ONE** A4 sheet of hand written notes is permitted.
5. Please use only the supplied answer books, and don't mix answers to different problems on the same sheet.
6. There is a table of Clebsch-Gordan coefficients attached.

1: Atomic Structure

(a) Explain the difference between LS and jj coupling in describing the level structure of multi-electron atoms.

(b) The following table gives the electronic configurations and energies (in cm^{-1}) for excited states of Lu^+ (relative to the $6s^2$ ground state).

Config.	Energy
6s5d	11796.24
	12435.32
	14199.08
	17332.58
6s6p	27264.40
	28503.16
	32453.26
	38223.49

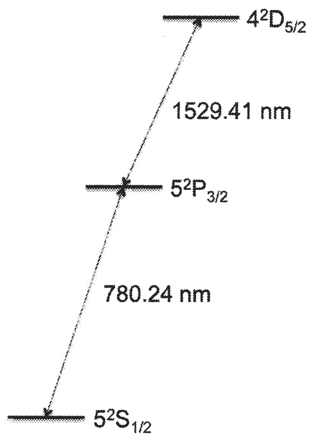
(i) Suggest, with reasons, further quantum numbers to identify these levels.

(ii) Draw an energy level diagram showing the allowed dipole transitions, within the LS coupling regime.

(iii) Explain why spin forbidden transitions appear in the spectra of some atoms. For the Lu^+ level structure given above, give a list of the possible spin forbidden transitions that may appear. Explain your reasoning.

— Please turn over —

2: Absorption Spectroscopy



The figure shows two transitions of interest for ^{87}Rb and their associated resonant wavelengths. The Einstein A coefficients for the decays $5\,^2\mathrm{P}_{3/2} \rightarrow 5\,^2\mathrm{S}_{1/2}$ and $4\,^2\mathrm{D}_{5/2} \rightarrow 5\,^2\mathrm{P}_{3/2}$ are $2\pi \times 6.0\,\mathrm{MHz}$ and $2\pi \times 2.0\,\mathrm{MHz}$ respectively. Absorption spectroscopy of the transitions is performed by passing co-propagating and co-linear beams, one at 780.24 nm and one at 1529.41 nm, through a room temperature (300 K) vapour of ^{87}Rb. It what follows assume both beams are σ^+ polarised and ignore any hyperfine structure.

(a) Derive an expression for the absorption coefficient of the 780.24 nm laser as a function of its detuning, Δ_1, from the $5\,^2\mathrm{S}_{1/2}$ to $5\,^2\mathrm{P}_{3/2}$ resonance, the density of atoms n, and the resonant absorption cross-section σ_0. You may assume the laser intensity is much less than the saturation intensity of the transition. Justify any assumptions you make.

(b) Consider the case in which the detuning, Δ_1, is fixed at a value giving non-negligible absorption of the 780.24 nm laser light. Estimate the width and position of the absorption spectrum for the 1529.41nm laser as a function of its detuning, Δ_2 from the $5\,^2\mathrm{P}_{3/2}$ to $4\,^2\mathrm{D}_{5/2}$ transition. A detailed calculation is not needed but your reasoning should be clearly explained.

(c) The $4\,^2\mathrm{D}_{5/2}$ upper state can be probed directly via two photon spectroscopy with a laser at 1033.32 nm. *Briefly* explain the principle of two-photon spectroscopy explaining how Doppler broadening is eliminated in this approach. List factors that may contribute to the broadening and/or shifting of the resonant frequency when implementing this technique.

— Please turn over —

3: Hyperfine interaction and Magnetic fields

(a) The hyperfine structure the $2S_{1/2}$ ground state of an atom is described by the perturbation

$$H_{\text{hfs}} = \frac{2}{3}\frac{\omega_0}{\hbar}\mathbf{I}\cdot\mathbf{J}.$$

Show that the states

$$|\,F, m_F\,\rangle = |\,l, s, j, I, F, m_F\,\rangle$$

are eigenstates of the perturbation where $\mathbf{F} = \mathbf{J} + \mathbf{I}$. Taking $I = 1$, show that the hyperfine splitting is given by $\hbar\omega_0$.

(b) In a magnetic field, the Zeeman interaction is described by the perturbation

$$H_Z = \frac{\mu_B}{\hbar}(\mathbf{L} + 2\mathbf{S})\cdot\mathbf{B}$$

where we have neglected the small contribution from the nuclear moment and taken $g_s = 2$. Determine the effect of a static B-field $\mathbf{B} = B_0\hat{\mathbf{z}}$ in the limit that $\mu_B B_0 \ll \hbar\omega_0$.

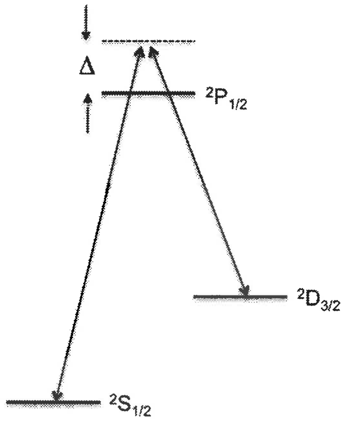
(c) In addition to the field given in (b), a time dependent field

$$\mathbf{B} = B_{rf}(\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}})$$

is used to invoke transitions between hyperfine states. For an atom prepared in the state $|\,F = 1/2, m_F = -1/2\,\rangle$, determine all possible transitions this field can produce. For each transition, stipulate the states involved, and the resonant frequency ω. You may assume $\omega_0, B_0 > 0$.

— Please turn over —

4: Raman transitions and AC Stark shifts



The figure shows the energy level diagram for the $6^2S_{1/2}$, $6^2P_{1/2}$, and $5^2D_{3/2}$ levels of $^{138}Ba^+$. The $6^2P_{1/2}$ and $5^2D_{3/2}$ levels are at $20261.561\,cm^{-1}$ and $4873.852\,cm^{-1}$ respectively from the $6^2S_{1/2}$ ground state. The decay rates from $6^2P_{1/2}$ to $6^2S_{1/2}$ and from $6^2P_{1/2}$ to $5^2D_{3/2}$ are $2\pi \times 15.427$ MHz and $2\pi \times 5.276$ MHz respectively. A two photon Raman transition is used to transfer population from $6^2S_{1/2}$ to $5^2D_{3/2}$ as indicated. The intensity of each laser field is $10\,MWm^{-2}$ and the detuning $\Delta = 2\pi \times 500$ GHz.

(a) Assuming the beams are co-propagating, suggest polarisations for each beam to transfer population from $|\,^2S_{1/2}, m_J = 1/2\,\rangle$ to $|\,^2D_{3/2}, m_J = 3/2\,\rangle$. Explain your choice.

(b) Calculate the Rabi rate (in MHz) of the two photon transition for the configuration you gave in (a).

(c) Calculate the induced AC Stark shifts (in MHz) of $|\,^2S_{1/2}, m_J = 1/2\,\rangle$ and $|\,^2D_{3/2}, m_J = 3/2\,\rangle$ due to the driving fields. Justify any assumptions you make.

(d) Will the configuration under consideration give any coupling to the motion of the ion? Explain.

Note: You may find the following equations useful

$$I_0 = \frac{1}{2}\,\epsilon_0 c E_0^2, \qquad A_{ij} = \frac{\omega_{ij}^3\,\mu_{ij}^2}{3\pi\epsilon_0\hbar c^3}$$

$$\epsilon_0 = 8.85 \times 10^{-12}\,\text{F/m}, \quad c = 2.9979 \times 10^8\,\text{m/s}, \quad \hbar = 1.055 \times 10^{-34}\,\text{Js}$$

[MDB]

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36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2			
m_1 m_2	Coefficients		
. .			

1/2×1/2

m_1 m_2	1	0
M	+1	
+1/2 +1/2	1	
M	0	0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2
M	−1	
−1/2 −1/2	1	

1×1/2

m_1 m_2	3/2	1/2
M	+3/2	
+1 +1/2	1	
M	+1/2	+1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3
M	−1/2	−1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3
M	−3/2	
−1 −1/2	1	

2×1

m_1 m_2	3	2	1
M	+3		
+2 +1	1		
M	+2	+2	
+2 0	1/3	2/3	
+1 +1	2/3	−1/3	
M	+1	+1	+1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10
M	0	0	0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10
M	−1	−1	−1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5
M	−2	−2	
−1 −1	2/3	1/3	
−2 0	1/3	−2/3	
M	−3		
−2 −1	1		

1×1

m_1 m_2	2	1	0
M	+2		
+1 +1	1		
M	+1	+1	
+1 0	1/2	1/2	
0 +1	1/2	−1/2	
M	0	0	0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3
M	−1	−1	
0 −1	1/2	1/2	
−1 0	1/2	−1/2	
M	−2		
−1 −1	1		

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

2×1/2

m_1 m_2	5/2	3/2
M	+5/2	
+2 +1/2	1	
M	+3/2	+3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5
M	+1/2	+1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5
M	−1/2	−1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5
M	−3/2	−3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5
M	−5/2	
−2 −1/2	1	

3/2×1/2

m_1 m_2	2	1
M	+2	
+3/2 +1/2	1	
M	+1	+1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4
M	0	0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2
M	−1	−1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4
M	−2	
−3/2 −1/2	1	

3/2×1

m_1 m_2	5/2	3/2	1/2
M	+5/2		
+3/2 +1	1		
M	+3/2	+3/2	
+3/2 0	2/5	3/5	
+1/2 +1	3/5	−2/5	
M	+1/2	+1/2	+1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6
M	−1/2	−1/2	−1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2
M	−3/2	−3/2	
−1/2 −1	3/5	2/5	
−3/2 0	2/5	−3/5	
M	−5/2		
−3/2 −1	1		

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

3/2×3/2

m_1 m_2	3	2	1	0
M	+3			
+3/2 +3/2	1			
M	+2	+2		
+3/2 +1/2	1/2	1/2		
+1/2 +3/2	1/2	−1/2		
M	+1	+1	+1	
+3/2 −1/2	1/5	1/2	3/10	
+1/2 +1/2	3/5	0	−2/5	
−1/2 +3/2	1/5	−1/2	3/10	
M	0	0	0	0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4
M	−1	−1	−1	
+1/2 −3/2	1/5	1/2	3/10	
−1/2 −1/2	3/5	0	−2/5	
−3/2 +1/2	1/5	−1/2	3/10	
M	−2	−2		
−1/2 −3/2	1/2	1/2		
−3/2 −1/2	1/2	−1/2		
M	−3			
−3/2 −3/2	1			

2×3/2

m_1 m_2	7/2	5/2	3/2	1/2
M	+7/2			
+2 +3/2	1			
M	+5/2	+5/2		
+2 +1/2	3/7	4/7		
+1 +3/2	4/7	−3/7		
M	+3/2	+3/2	+3/2	
+2 −1/2	1/7	16/35	2/5	
+1 +1/2	4/7	1/35	−2/5	
0 +3/2	2/7	−18/35	1/5	
M	+1/2	+1/2	+1/2	+1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10
M	−1/2	−1/2	−1/2	−1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5
M	−3/2	−3/2	−3/2	
0 −3/2	2/7	18/35	1/5	
−1 −1/2	4/7	−1/35	−2/5	
−2 +1/2	1/7	−16/35	2/5	
M	−5/2	−5/2		
−1 −3/2	4/7	3/7		
−2 −1/2	3/7	−4/7		
M	−7/2			
−2 −3/2	1			

2×2

m_1 m_2	4	3	2	1	0
M	+4				
+2 +2	1				
M	+3	+3			
+2 +1	1/2	1/2			
+1 +2	1/2	−1/2			
M	+2	+2	+2		
+2 0	3/14	1/2	2/7		
+1 +1	4/7	0	−3/7		
0 +2	3/14	−1/2	2/7		
M	+1	+1	+1	+1	
+2 −1	1/14	3/10	3/7	1/5	
+1 0	3/7	1/5	−1/14	−3/10	
0 +1	3/7	−1/5	−1/14	3/10	
−1 +2	1/14	−3/10	3/7	−1/5	
M	0	0	0	0	0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5
M	−1	−1	−1	−1	
+1 −2	1/14	3/10	3/7	1/5	
0 −1	3/7	1/5	−1/14	−3/10	
−1 0	3/7	−1/5	−1/14	3/10	
−2 +1	1/14	−3/10	3/7	−1/5	
M	−2	−2	−2		
0 −2	3/14	1/2	2/7		
−1 −1	4/7	0	−3/7		
−2 0	3/14	−1/2	2/7		
M	−3	−3			
−1 −2	1/2	1/2			
−2 −1	1/2	−1/2			
M	−4				
−2 −2	1				

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2 \qquad d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta \qquad d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^2_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1) \qquad d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta \qquad d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).