NATIONAL UNIVERSITY OF SINGAPORE

PC4243: Atomic & Molecular Physics II

(Semester 2: AY 2014-15)

Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. Write your matriculation number only on your answer booklet. Do not write your name.
2. This exam paper contains **FOUR** questions and comprises **SIX** printed pages.
3. You have to answer **THREE** out of the four questions.
4. Non-programmable calculators are permitted.
5. This is a **CLOSED BOOK** examination but **ONE** A4 sheet of hand written notes is permitted.
6. Please use only the supplied answer books, and don't mix answers to different problems on the same sheet.
7. There is a table of Clebsch-Gordan coefficients attached.

1: Atomic Structure

(a) Explain the difference between LS and jj coupling in describing the level structure of multi-electron atoms.

(b) The following table gives the electronic configurations and energies (in cm^{-1}) for excited states of neutral Yb (relative to the 6s^2 ground state).

Config.	Energy
6s6p	17288.439
	17992.007
	19710.388
	25068.222
6s5d	24489.102
	24751.948
	25270.902
	27677.655

(i) Suggest, with reasons, further quantum numbers to identify these levels.

(ii) Draw an energy level diagram showing the allowed dipole transitions, within the LS coupling regime.

(iii) Explain why spin forbidden transitions appear in the spectra of some atoms. For the Yb level structure given above, give a list of the possible spin forbidden transitions that may appear. Explain your reasoning.

(iv) The Einstein A coefficients for decay from the 17992.007 and 25270.902 levels to the ground state are $1.265\times10^6\text{s}^{-1}$ and $8.918\times10^7\text{s}^{-1}$ respectively. Give an order of magnitude estimate for the Einstein A coefficient for decay from the 25270.902 level to the 24489.102 level.

(v) Which transitions in neutral Yb would be suitable for making a magneto-optic trap. Of these transitions which would likely yield the highest number of atoms? or the lowest temperature? Explain your reasoning.

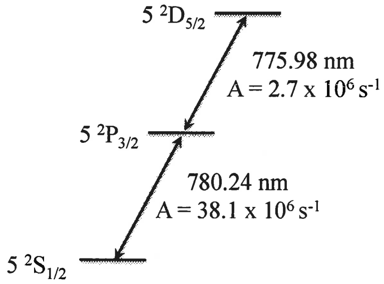
Note: You may find the following helpful

$$A_{ij} = \frac{\omega_{ij}^3\,\mu_{ij}^2}{3\pi\epsilon_0\hbar c^3}, \quad a_0 = 5.292\times10^{-11}\,\text{m}$$

$$\epsilon_0 = 8.85\times10^{-12}\,\text{F/m}, \quad c = 2.9979\times10^8\,\text{m/s}, \quad \hbar = 1.055\times10^{-34}\,\text{Js}$$

— Please turn over —

2: Two Photon Spectroscopy



The figure shows two transitions of interest for ^{87}Rb, which has nuclear spin $I = 3/2$. Also shown is the resonant wavelengths and Einstein A coefficients for the two transitions. The total line-width of the $5\,^2\mathrm{D}_{5/2}$ level is $\Gamma = 2\pi \times 600\,\mathrm{kHz}$ due to additional decays not included in the figure. The $5\,^2\mathrm{S}_{1/2}$ to $5\,^2\mathrm{D}_{5/2}$ two-photon transition is driven with a single retro-reflected σ^+ polarised laser field which resonantly couples the $|\,F = 2, m_F = 2\,\rangle$ ground state to the upper $|\,F = 4, m_F = 4\,\rangle$ state.

(a) *Briefly* describe the principle of two-photon spectroscopy explaining how Doppler broadening is eliminated in this approach. List other mechanisms that may shift/broaden the line.

(b) Determine the laser intensity needed such that the two-photon Rabi frequency is equal to the total line-width of the upper $5\,^2\mathrm{D}_{5/2}$ state.

(c) For the laser intensity found in (b), determine the total AC stark shift of the two-photon transition.

(d) The effect of a magnetic field is well described by the Zeeman interaction $H = \mu_B \mathbf{B} \cdot (\mathbf{L} + 2\mathbf{S})/\hbar$. Determine the magnetic field such that the Zeeman shift of the two-photon transition equals the total line-width of the upper state.

(e) A small angle, θ, between the reflected and incoming beam results in a residual Doppler shift. Determine the angle such that the rms Doppler broadening equals the total line-width of the upper state for an atomic vapour at a temperature of 400 K.

Note: You may find the following helpful

$$I_0 = \frac{1}{2}\,\epsilon_0 c E_0^2, \qquad A_{ij} = \frac{\omega_{ij}^3\,\mu_{ij}^2}{3\pi\epsilon_0\hbar c^3}$$

$$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{F/m}, \quad c = 2.9979 \times 10^8\,\mathrm{m/s}, \quad \hbar = 1.055 \times 10^{-34}\,\mathrm{Js}$$

— Please turn over —

3: Hyperfine interaction and Magnetic fields

The hyperfine structure for a $^3\mathrm{D}_1$ metastable state of an atom with nuclear spin $I = 2$ is described by

$$H_{\mathrm{hfs}} = \frac{\omega_0}{\hbar}\mathbf{I}\cdot\mathbf{J}.$$

Additionally the effects of a magnetic field are described by

$$H_Z = \frac{\mu_B}{\hbar}(\mathbf{L} + 2\mathbf{S})\cdot\mathbf{B},$$

where we have neglected the small contribution from the nuclear moment and taken $g_s = 2$.

(a) What is the physical origin of the interaction that leads to hyperfine structure in atoms?

(b) Show that the states

$$|\,F, m_F\,\rangle = |\,l, s, j, I, F, m_F\,\rangle$$

are eigenstates of H_{hfs} where $\mathbf{F} = \mathbf{J} + \mathbf{I}$ and find the hyperfine splittings in terms of ω_0.

(c) Determine the effect of a static B-field $\mathbf{B} = B_0\hat{\mathbf{z}}$ in the limit that $\mu_B B_0 \ll \hbar\omega_0$.

(d) In addition to the field given in (c), a time dependent field

$$\mathbf{B} = B_{rf}(\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}})$$

is used to invoke transitions between hyperfine states. For an atom prepared in the state $|\,F = 2, m_F = 0\,\rangle$, determine all possible transitions this field can produce. For each transition, stipulate the states involved, and the resonant frequency ω. You may assume $\omega_0, B_0 > 0$.

Hint: It is will be useful to express the time-dependent Hamiltonian in terms of $\mathbf{J}$ and use $\mathbf{J}_\pm = \mathbf{J}_x \pm i\mathbf{J}_y$.

— Please turn over —

4: Laser Cooling

In steady state, the excited state population for a two level atom is given by

$$\rho_{ee} = \frac{1}{2}\frac{I/I_s}{1 + I/I_s + 4\Delta^2/\Gamma^2}$$

where I_s is the saturation intensity, Δ is the laser detuning from the atomic resonance, and Γ is the line-width of the excited state.

(a) Explain how photon scattering results in a force on an atom. Give an expression for the scattering force on a stationary atom.

(b) A Mg^+ ion is confined by an harmonic potential with trapping frequency $\omega_T = 2\pi \times 400$ kHz. The ion experiences a scattering force from laser light of wavelength, $\lambda = 280$ nm, and intensity, I, which excites a transition with line-width $\Gamma = 2\pi \times 43$ MHz. Show that the ion undergoes damped harmonic motion. Derive an expression for the damping coefficient, α, and static displacement, z_0, due to the laser beam for a given Δ/Γ and I/I_s.

(c) What is the physical mechanism that limits the final temperature in Doppler cooling?

(d) Assuming that the scattering in (b) is isotropic, give an account of the heating along the direction of the laser and hence derive an expression for the final temperature of the ion for a given Δ/Γ and I/I_s.

[MDB]

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36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J M	J M	… …
m_1	m_2	Coefficients		
m_1	m_2			
⋮	⋮			

1/2×1/2

		1 +1
+1/2	+1/2	1

		1 0	0 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1 −1
−1/2	−1/2	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

2×1/2

		5/2 +5/2
+2	+1/2	1

		5/2 +3/2	3/2 +3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

		5/2 +1/2	3/2 +1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

		5/2 −1/2	3/2 −1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

		5/2 −3/2	3/2 −3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

		5/2 −5/2
−2	−1/2	1

1×1/2

		3/2 +3/2
+1	+1/2	1

		3/2 +1/2	1/2 +1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2 −1/2	1/2 −1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2 −3/2
−1	−1/2	1

3/2×1/2

		2 +2
+3/2	+1/2	1

		2 +1	1 +1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

		2 0	1 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		2 −1	1 −1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

		2 −2
−3/2	−1/2	1

2×1

		3 +3
+2	+1	1

		3 +2	2 +2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

		3 +1	2 +1	1 +1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

		3 0	2 0	1 0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

		3 −1	2 −1	1 −1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

		3 −2	2 −2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

		3 −3
−2	−1	1

3/2×1

		5/2 +5/2
+3/2	+1	1

		5/2 +3/2	3/2 +3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

		5/2 +1/2	3/2 +1/2	1/2 +1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

		5/2 −1/2	3/2 −1/2	1/2 −1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

		5/2 −3/2	3/2 −3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

		5/2 −5/2
−3/2	−1	1

1×1

		2 +2
+1	+1	1

		2 +1	1 +1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2 0	1 0	0 0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2 −1	1 −1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2 −2
−1	−1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\,Y_\ell^m\,e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

3/2×3/2

		3 +3
+3/2	+3/2	1

		3 +2	2 +2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

		3 +1	2 +1	1 +1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

		3 0	2 0	1 0	0 0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

		3 −1	2 −1	1 −1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

		3 −2	2 −2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

		3 −3
−3/2	−3/2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1}_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

		7/2 +7/2
+2	+3/2	1

		7/2 +5/2	5/2 +5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

		7/2 +3/2	5/2 +3/2	3/2 +3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

		7/2 +1/2	5/2 +1/2	3/2 +1/2	1/2 +1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

		7/2 −1/2	5/2 −1/2	3/2 −1/2	1/2 −1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

		7/2 −3/2	5/2 −3/2	3/2 −3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

		7/2 −5/2	5/2 −5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

		7/2 −7/2
−2	−3/2	1

2×2

		4 +4
+2	+2	1

		4 +3	3 +3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

		4 +2	3 +2	2 +2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

		4 +1	3 +1	2 +1	1 +1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

		4 0	3 0	2 0	1 0	0 0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

		4 −1	3 −1	2 −1	1 −1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

		4 −2	3 −2	2 −2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

		4 −3	3 −3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

		4 −4
−2	−2	1

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\,\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).