NATIONAL UNIVERSITY OF SINGAPORE

PC4243: Atomic & Molecular Physics II

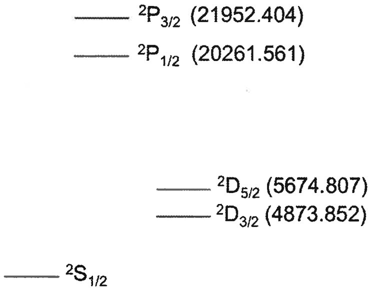
(Semester 2: AY 2016-17)

Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. Write your matriculation number only on your answer booklet. Do not write your name.
2. This exam paper contains **FOUR** questions and comprises **SIX** printed pages.
3. You have to answer **THREE** out of the four questions.
4. Non-programmable calculators are permitted.
5. This is a **CLOSED BOOK** examination but **ONE** A4 sheet of hand written notes is permitted.
6. Please use only the supplied answer books, and don't mix answers to different problems on the same sheet.
7. There is a table of Clebsch-Gordan coefficients attached.

1: Atomic Structure



The figure above shows the level structure of singly-ionized Barium (Ba^+) for the three lowest electronic configurations. The numbers in parentheses give the energies in cm^{-1} of each level relative to the $^2S_{1/2}$ ground state. The lifetime of $^2P_{3/2}$ is measured to be 6.4 ns and the branching ratio for decay to $^2D_{5/2}$ is measured to be 0.215.

(a) List all possible states that can be populated from an allowed electric dipole decay of $|\,^2P_{3/2}, m_J = 3/2\,\rangle$. For each state, give an expansion in the LS basis assuming *LS*-coupling holds.

(b) For each possible decay from $|\,^2P_{3/2}, m_J = 3/2\,\rangle$, express the decay rate in terms of the reduced dipole matrix elements $\langle\, P\,||r||\,S\,\rangle$ and $\langle\, P\,||r||\,D\,\rangle$.

(c) From your answer in (b), estimate all branching ratios for decays to the lower S and D levels from either P level. Also determine the lifetime of $^2P_{1/2}$.

Note: You may find the following helpful

$$A_{ij} = \frac{\omega_{ij}^3\,\mu_{ij}^2}{3\pi\epsilon_0\hbar c^3},$$

$$\epsilon_0 = 8.85\times10^{-12}\,\mathrm{F\,m^{-1}}, \quad c = 2.9979\times10^{8}\,\mathrm{m\,s^{-1}}, \quad \hbar = 1.055\times10^{-34}\,\mathrm{J\,s}$$

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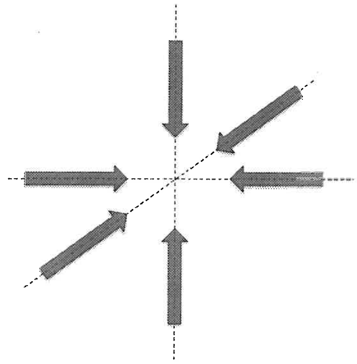
2: Photon recoil



(a) What is the physical mechanism that limits the final temperature when Doppler cooling two level atoms?

(b) Consider an ensemble of two level atoms that are initially at rest ($k_BT = 0$). The atoms are excited by three pairs of counter-propagating beams along three mutually orthogonal directions as illustrated in the figure above. The duration of the excitation is such that the atoms scatter one photon per atom on average.

(i) Determine the energy gained by the ensemble and hence the final temperature. Express your answer in terms of the recoil temperature (T_r) defined by $k_BT_r = (\hbar k)^2/2m$ where k is the wavenumber of the light field and m is the mass of the atom.

(ii) What would be the mean and root-mean-square (rms) velocity of the atoms along each direction? Express your answer in terms of the recoil velocity ($v_r = \hbar k/m$).

(iii) How would you answer to (ii) change if only one of the six beams was used, assuming that the atoms still scatter one photon per atom on average?

(iv) How would you answer to (ii) change if the atoms were excited with a π-pulse by one of the six beams and subsequently decayed? Assume the π-pulse is much shorter than the lifetime of the excited state.

3: AC Stark shifts and the Zeeman effect

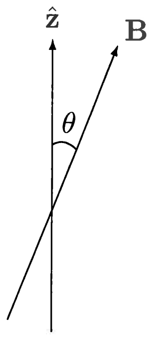
Consider the $^2S_{1/2}$ to $^2P_{3/2}$ transition of ^{87}Rb, which has a nuclear spin $I = 3/2$. An atom is subject to a laser field propagating along the positive z-axis with electric field amplitude E_0, polarisation $\hat{\mathbf{e}}_+ = -(\hat{\mathbf{x}} + i\hat{\mathbf{y}})/\sqrt{2}$, and detuning Δ from resonance with the $^2S_{1/2}$ to $^2P_{3/2}$ transition. In what follows you may assume Δ is much larger than the $^2P_{3/2}$ hyperfine splittings and neglect any coupling to $^2P_{1/2}$.

(a) Determine the AC Stark shifts for all $F = 1$ ground states in terms of Δ, E_0, and the reduced dipole matrix element $\mu = \langle\, J' = 3/2\, ||r||\, J = 1/2\,\rangle$.

(b) Show that the AC Stark shifts found in (a) can, up to a constant term, be described by an effective magnetic field. **Note:** $g_F = -1/2$ for $F = 1$.

(c) Suppose that the atom is prepared in $|\, F = 1, m_F = 1\,\rangle$. In addition to the laser field, a static magnetic field $\mathbf{B} = B_0\,(\cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\mathbf{x}})$ is applied at an oblique angle, θ, to the z-axis as shown below. Show that there is a B_0 at which the atom will oscillate between the $m_F = \pm 1$ states. Comment on any practical implications this may have for Sisyphus cooling.

Note: $J\pm = J_x \pm iJ_y, \quad J_\pm|\, j, m\,\rangle = \sqrt{j(j+1) \mp m(m \pm 1)}|\, j, m \pm 1\,\rangle$



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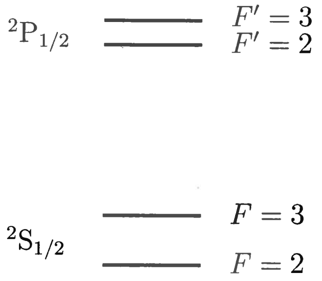
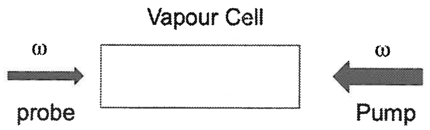
4: Saturated absorption spectroscopy





The figure above shows the hyperfine structure of the $^2S_{1/2}$ to $^2P_{1/2}$ transition of ^{85}Rb at 795 nm. The natural linewidth of the transition is $\Gamma = 2\pi \times 5.75$ MHz. The hyperfine splittings of $^2S_{1/2}$ and $^2P_{1/2}$ are 3 GHz and 360 MHz, respectively. Doppler free saturated absorption spectroscopy is carried out on the transition with a set up shown schematically to the right.

(a) Calculate the amount of Doppler broadening at room temperature (300 K) for this transition. Give a brief description or illustration of the expected probe transmission as a function of laser frequency.

(b) Consider the absorption feature near to the $F = 3$ to $F' = 3$ transition. Sketch the velocity distributions for the $F = 3$ ground state when the lasers are tuned to (i) $\omega - \omega_0 = 2\pi \times 90$ MHz, (ii) $\omega - \omega_0 = -2\pi \times 90$ MHz, and (iii) $\omega - \omega_0 = -2\pi \times 180$ MHz where ω_0 is the resonant frequency of the $F = 3$ to $F' = 3$ transition. Your sketch need not be to scale, but you should comment on the width, relative size, and relative position of any features you include. You may neglect scattering into the $F = 2$ ground state.

(c) With reference to the diagrams in (b) or otherwise, give a brief account of Doppler free saturated absorption spectroscopy. No calculations are required.

(d) Briefly discuss the influence that optical pumping and polarization can have on probe transmission.

Note:

$$k_B = 1.38 \times 10^{-23}\,\text{J/K}, \quad 1\,\text{a.m.u.} = 1.66 \times 10^{-27}\,\text{kg}$$

[MDB]

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36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2	Coefficients		
m_1 m_2			
. .			
. .			

1/2×1/2

	1
	+1
+1/2 +1/2	1

	1	0
	0	0
+1/2 -1/2	1/2	1/2
-1/2 +1/2	1/2	-1/2

	1
	-1
-1/2 -1/2	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

2×1/2

	5/2
	+5/2
+2 +1/2	1

	5/2	3/2
	+3/2	+3/2
+2 -1/2	1/5	4/5
+1 +1/2	4/5	-1/5

	5/2	3/2
	+1/2	+1/2
+1 -1/2	2/5	3/5
0 +1/2	3/5	-2/5

	5/2	3/2
	-1/2	-1/2
0 -1/2	3/5	2/5
-1 +1/2	2/5	-3/5

	5/2	3/2
	-3/2	-3/2
-1 -1/2	4/5	1/5
-2 +1/2	1/5	-4/5

	5/2
	-5/2
-2 -1/2	1

1×1/2

	3/2
	+3/2
+1 +1/2	1

	3/2	1/2
	+1/2	+1/2
+1 -1/2	1/3	2/3
0 +1/2	2/3	-1/3

	3/2	1/2
	-1/2	-1/2
0 -1/2	2/3	1/3
-1 +1/2	1/3	-2/3

	3/2
	-3/2
-1 -1/2	1

3/2×1/2

	2
	+2
+3/2 +1/2	1

	2	1
	+1	+1
+3/2 -1/2	1/4	3/4
+1/2 +1/2	3/4	-1/4

	2	1
	0	0
+1/2 -1/2	1/2	1/2
-1/2 +1/2	1/2	-1/2

	2	1
	-1	-1
-1/2 -1/2	3/4	1/4
-3/2 +1/2	1/4	-3/4

	2
	-2
-3/2 -1/2	1

2×1

	3
	+3
+2 +1	1

	3	2
	+2	+2
+2 0	1/3	2/3
+1 +1	2/3	-1/3

	3	2	1
	+1	+1	+1
+2 -1	1/15	1/3	3/5
+1 0	8/15	1/6	-3/10
0 +1	2/5	-1/2	1/10

	3	2	1
	0	0	0
+1 -1	1/5	1/2	3/10
0 0	3/5	0	-2/5
-1 +1	1/5	-1/2	3/10

	3	2	1
	-1	-1	-1
0 -1	2/5	1/2	1/10
-1 0	8/15	-1/6	-3/10
-2 +1	1/15	-1/3	3/5

	3	2
	-2	-2
-1 -1	2/3	1/3
-2 0	1/3	-2/3

	3
	-3
-2 -1	1

3/2×1

	5/2
	+5/2
+3/2 +1	1

	5/2	3/2
	+3/2	+3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	-2/5

	5/2	3/2	1/2
	+1/2	+1/2	+1/2
+3/2 -1	1/10	2/5	1/2
+1/2 0	3/5	1/15	-1/3
-1/2 +1	3/10	-8/15	1/6

	5/2	3/2	1/2
	-1/2	-1/2	-1/2
+1/2 -1	3/10	8/15	1/6
-1/2 0	3/5	-1/15	-1/3
-3/2 +1	1/10	-2/5	1/2

	5/2	3/2
	-3/2	-3/2
-1/2 -1	3/5	2/5
-3/2 0	2/5	-3/5

	5/2
	-5/2
-3/2 -1	1

1×1

	2
	+2
+1 +1	1

	2	1
	+1	+1
+1 0	1/2	1/2
0 +1	1/2	-1/2

	2	1	0
	0	0	0
+1 -1	1/6	1/2	1/3
0 0	2/3	0	-1/3
-1 +1	1/6	-1/2	1/3

	2	1
	-1	-1
0 -1	1/2	1/2
-1 0	1/2	-1/2

	2
	-2
-1 -1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

3/2×3/2

	3
	+3
+3/2 +3/2	1

	3	2
	+2	+2
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	-1/2

	3	2	1
	+1	+1	+1
+3/2 -1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	-2/5
-1/2 +3/2	1/5	-1/2	3/10

	3	2	1	0
	0	0	0	0
+3/2 -3/2	1/20	1/4	9/20	1/4
+1/2 -1/2	9/20	1/4	-1/20	-1/4
-1/2 +1/2	9/20	-1/4	-1/20	1/4
-3/2 +3/2	1/20	-1/4	9/20	-1/4

	3	2	1
	-1	-1	-1
+1/2 -3/2	1/5	1/2	3/10
-1/2 -1/2	3/5	0	-2/5
-3/2 +1/2	1/5	-1/2	3/10

	3	2
	-2	-2
-1/2 -3/2	1/2	1/2
-3/2 -1/2	1/2	-1/2

	3
	-3
-3/2 -3/2	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

	7/2
	+7/2
+2 +3/2	1

	7/2	5/2
	+5/2	+5/2
+2 +1/2	3/7	4/7
+1 +3/2	4/7	-3/7

	7/2	5/2	3/2
	+3/2	+3/2	+3/2
+2 -1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	-2/5
0 +3/2	2/7	-18/35	1/5

	7/2	5/2	3/2	1/2
	+1/2	+1/2	+1/2	+1/2
+2 -3/2	1/35	6/35	2/5	2/5
+1 -1/2	12/35	5/14	0	-3/10
0 +1/2	18/35	-3/35	-1/5	1/5
-1 +3/2	4/35	-27/70	2/5	-1/10

	7/2	5/2	3/2	1/2
	-1/2	-1/2	-1/2	-1/2
+1 -3/2	4/35	27/70	2/5	1/10
0 -1/2	18/35	3/35	-1/5	-1/5
-1 +1/2	12/35	-5/14	0	3/10
-2 +3/2	1/35	-6/35	2/5	-2/5

	7/2	5/2	3/2
	-3/2	-3/2	-3/2
0 -3/2	2/7	18/35	1/5
-1 -1/2	4/7	-1/35	-2/5
-2 +1/2	1/7	-16/35	2/5

	7/2	5/2
	-5/2	-5/2
-1 -3/2	4/7	3/7
-2 -1/2	3/7	-4/7

	7/2
	-7/2
-2 -3/2	1

2×2

	4
	+4
+2 +2	1

	4	3
	+3	+3
+2 +1	1/2	1/2
+1 +2	1/2	-1/2

	4	3	2
	+2	+2	+2
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	-3/7
0 +2	3/14	-1/2	2/7

	4	3	2	1
	+1	+1	+1	+1
+2 -1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	-1/14	-3/10
0 +1	3/7	-1/5	-1/14	3/10
-1 +2	1/14	-3/10	3/7	-1/5

	4	3	2	1	0
	0	0	0	0	0
+2 -2	1/70	1/10	2/7	2/5	1/5
+1 -1	8/35	2/5	1/14	-1/10	-1/5
0 0	18/35	0	-2/7	0	1/5
-1 +1	8/35	-2/5	1/14	1/10	-1/5
-2 +2	1/70	-1/10	2/7	-2/5	1/5

	4	3	2	1
	-1	-1	-1	-1
+1 -2	1/14	3/10	3/7	1/5
0 -1	3/7	1/5	-1/14	-3/10
-1 0	3/7	-1/5	-1/14	3/10
-2 +1	1/14	-3/10	3/7	-1/5

	4	3	2
	-2	-2	-2
0 -2	3/14	1/2	2/7
-1 -1	4/7	0	-3/7
-2 0	3/14	-1/2	2/7

	4	3
	-3	-3
-1 -2	1/2	1/2
-2 -1	1/2	-1/2

	4
	-4
-2 -2	1

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\,\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).