Question 1 (a)

Let the momenta of the two photons be $\vec{p}_1$ and $\vec{p}_2$. For zero-mass photons, $E_1 = |\vec{p}_1|c$ and $E_1 = |\vec{p}_2|c$.

$$\vec{p}_1 \cdot \vec{p}_2 = |\vec{p}_1||\vec{p}_2| \cos\theta = \frac{E_1 E_2 \cos\theta}{c^2}$$

Total 4-momentum of the photon pair $= \left(\frac{E_1+E_2}{c}, \vec{p}_1 + \vec{p}_2\right)$

Invariant mass of final state is

$$M^2c^2 = \frac{(E_1+E_2)^2}{c^2} - (\vec{p}_1 + \vec{p}_2)^2 = \frac{2E_1E_2(1-\cos\theta)}{c^2}$$

$$M^2 = \frac{4E_1E_2}{c^4}\sin^2\frac{\theta}{2}$$

Question 1 (b)

Conservation of energy: $E = E_1 + E_2$. Let

$$f(\theta) = \sin^2\frac{\theta}{2} = \frac{m^2c^4}{4E_1(E-E_1)}$$

Treat E_1 as variable and minimize f. Easy to show that f has a minimum at $E_1 = \frac{E}{2}$.

$$\text{Hence } \sin\frac{\theta_{\min}}{2} = \frac{mc^2}{E}$$

$$\text{But } \sin\frac{\theta}{2} \geq \frac{mc^2}{E} \quad \Rightarrow \quad mc^2 \leq E\sin\frac{\theta}{2}.$$

With $E = 10\text{GeV}$ and $\theta = 2°$, $mc^2 < 174.5\text{MeV}$.

Hence the particle can only be a π^0 meson, and cannot be a η meson.

Question 1 (c)

$$\Lambda^0 \rightarrow \gamma\gamma \qquad d\Gamma = \frac{S}{2\hbar m_1}|M|^2 \frac{d^3P_2}{2(2\pi)^3P_2^0}\frac{d^3P_3}{2(2\pi)^3P_3^0}(2\pi)^4\delta^{(4)}(P_1 - P_2 -$$

$$S = \frac{1}{2!} = \frac{1}{2}.$$

$$d\Gamma = \int \frac{|M|^2}{(8\pi)^2\hbar m_1}\frac{d^3\vec{P}_2}{|P_2|}\frac{d^3\vec{P}_3}{|P_3|}\delta^{(1)}(m_1c - |P_2| - |P_3|)\delta^{(3)}(-\vec{P}_2 - \vec{P}$$

$$= \int \frac{|M|^2}{(8\pi)^2\hbar m_1}\frac{|P_2|^2 d\vec{P}_2\, d\Omega}{|P_2|^2}\delta^{(1)}(m_1c - 2|P_2|)$$

$$= \frac{4\pi|M|^2}{(8\pi)^2\hbar m_1} = \text{ii)}\ \frac{|M|^2}{16\pi\hbar m_\Lambda} \qquad \text{i)}\ |P_2| = |P_3| = \frac{m_1c}{2}$$

Question 1 (d)

i) $e^+e^- \to \gamma\gamma$ (para-positronium, $S=0$)

ii) $e^+e^- \to \gamma\gamma\gamma$ (ortho-positronium, $S=1$)

$C = (-1)^{L+S}$

i) $C_i = (-1)^0 = 1 \qquad C_f = (-1)^2 = 1$

ii) $C_i = (-1)^{0+1} = -1 \qquad C_f = (-1)^3 = -1$

} charge conjugation invariance

$n=1$, disallowed from conservation of momentum.

CM frame : $\vec{p} = 0$ if one γ, $\vec{p} \neq 0$

Question 2 (a)

So $u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_z}{E+mc^2} \\ \dfrac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}$ & $u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x-ip_y)}{E+mc^2} \\ \dfrac{-cp_z}{E+mc^2} \end{pmatrix}$

are solutions, N some constant. Need to find solutions which are eigenspinors of $\frac{\Sigma\cdot\vec{p}}{|\vec{p}|}$. These are $u^{(+)}$ and $u^{(-)}$. Let $u^{(\pm)} = M\left(au^{(1)} + bu^{(2)}\right)$, M some constant. Then we require that:

$$\frac{\Sigma\cdot\vec{p}}{|\vec{p}|}u^{(\pm)} = \pm u^{(\pm)}, \quad \text{to find } a \,\&\, b. \text{ Now}$$

$$\frac{\Sigma\cdot\vec{p}}{|\vec{p}|} = \frac{1}{|\vec{p}|}\begin{pmatrix} \vec{\sigma}\cdot\vec{p} & \vec{0} \\ \vec{0} & \vec{\sigma}\cdot\vec{p} \end{pmatrix} = \frac{1}{|\vec{p}|}\begin{pmatrix} \begin{matrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{matrix} & \vec{0} \\ \vec{0} & \begin{matrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{matrix} \end{pmatrix}. \text{ So}$$

$$\frac{\Sigma\cdot\vec{p}}{|\vec{p}|}u^{(1)} = \frac{N}{|\vec{p}|}\begin{pmatrix} p_z \\ p_x + ip_y \\ \dfrac{c|\vec{p}|^2}{E+mc^2} \\ 0 \end{pmatrix}$$

$$\frac{\Sigma\cdot\vec{p}}{|\vec{p}|}u^{(2)} = \frac{N}{|\vec{p}|}\begin{pmatrix} p_x - ip_y \\ -p_z \\ 0 \\ \dfrac{c|\vec{p}|^2}{E+mc^2} \end{pmatrix}$$

Recap: We need

$$\frac{\Sigma\cdot\vec{p}}{|\vec{p}|}\left(au^{(1)} + bu^{(2)}\right) = \pm\left(au^{(1)} + bu^{(2)}\right)$$

$$\left(a\frac{N}{|\vec{p}|}\begin{pmatrix} p_z \\ p_x+ip_y \\ \dfrac{c|\vec{p}|^2}{E+mc^2} \\ 0\end{pmatrix} + b\frac{N}{|\vec{p}|}\begin{pmatrix} p_x-ip_y\\ -p_z \\ 0 \\ \dfrac{c|\vec{p}|^2}{E+mc^2}\end{pmatrix}\right) = \pm\left(aN\begin{pmatrix}1\\0\\ \dfrac{cp_z}{E+mc^2}\\ \dfrac{c(p_x+ip_y)}{E+mc^2}\end{pmatrix} + bN\begin{pmatrix}0\\1\\ \dfrac{c(p_x-ip_y)}{E+mc^2} \\ \dfrac{-cp_z}{E+mc^2}\end{pmatrix}\right)$$

$$\begin{pmatrix} \dfrac{1}{|\vec{p}|}\left[ap_z+b(p_x-ip_y)\right] \\ \dfrac{1}{|\vec{p}|}\left[a(p_x+ip_y)-bp_z\right] \\ \dfrac{ac|\vec{p}|}{E+mc^2} \\ \dfrac{bc|\vec{p}|}{E+mc^2}\end{pmatrix} = \pm\begin{pmatrix} a \\ b \\ \dfrac{acp_z+bc(p_x-ip_y)}{E+mc^2} \\ \dfrac{ac(p_x+ip_y)-bcp_z}{E+mc^2}\end{pmatrix}$$

$$\Rightarrow b(p_x-ip_y) = a(\pm|\vec{p}|-p_z) \;\; \& \;\; a(p_x+ip_y) = b(\pm|\vec{p}|+p_z)$$

If we choose $a = \pm|\vec{p}|+p_z$, then $b = p_x+ip_y$! So

$$u^{(\pm)} = M\begin{pmatrix} \pm|\vec{p}|+p_z \\ p_x+ip_y \\ \dfrac{c|\vec{p}|}{E+mc^2}(|\vec{p}|\pm p_z) \\ \pm\dfrac{c|\vec{p}|}{E+mc^2}(p_x+ip_y)\end{pmatrix}$$

Now $u^{(\pm)\dagger}u^{(\pm)} = |M|^2\left[(\pm|\vec{p}|+p_z)^2+p_x^2+p_y^2\right]\left[1+\dfrac{c^2|\vec{p}|^2}{(E+mc^2)^2}\right] = \dfrac{2E}{c}$

$$\Rightarrow |M| = \sqrt{\frac{E+mc^2}{2|\vec{p}|c(|\vec{p}|\pm p_z)}}$$

$$\therefore u^{(\pm)} = \sqrt{\frac{E+mc^2}{2|\vec{p}|c(|\vec{p}|\pm p_z)}}\begin{pmatrix} \pm|\vec{p}|+p_z \\ p_x+ip_y \\ \dfrac{c|\vec{p}|}{E+mc^2}(|\vec{p}|\pm p_z) \\ \pm\dfrac{c|\vec{p}|}{E+mc^2}(p_x+ip_y)\end{pmatrix}$$

Question 2 (b) (i)

$$\gamma^\mu\gamma^\nu(1-\gamma^5)\gamma^\lambda(1+\gamma^5)\gamma_\lambda$$

$$= \gamma^\mu\gamma^\nu(1-\gamma^5)^2\gamma^\lambda\gamma_\lambda$$

$$= 2(4)\gamma^\mu\gamma^\nu(1-\gamma^5) = 8\gamma^\mu\gamma^\nu(1-\gamma^5)$$

Question 2 (b) (ii)

$\mathrm{Tr}[\gamma^{\lambda_1}\gamma^{\lambda_2}\dots\gamma^{\lambda_n}] = \mathrm{Tr}[\gamma^{\lambda_1}\gamma^{\lambda_2}\dots\gamma^{\lambda_n}\gamma^5\gamma^5]$

$\lambda = \{1, 2, 3, 0\}$, since $\{\gamma^\lambda, \gamma^5\} = 0$

$= (-1)^n \mathrm{Tr}[\gamma^5\gamma^{\lambda_1}\gamma^{\lambda_2}\dots\gamma^{\lambda_n}\gamma^5]$

$= (-1)^n \mathrm{Tr}[\gamma^{\lambda_1}\gamma^{\lambda_2}\dots\gamma^{\lambda_n}\gamma^5\gamma^5]$

by $\mathrm{Tr}[AB] = \mathrm{Tr}[BA]$

$\mathrm{Tr}[\gamma^{\lambda_1}\gamma^{\lambda_2}\dots\gamma^{\lambda_n}] = (-1)^n \mathrm{Tr}[\gamma^{\lambda_1}\gamma^{\lambda_2}\dots\gamma^{\lambda_n}]$

$\therefore$ if n = odd, $\mathrm{Tr}[\ ] = 0$

Question 2 (b) (iii)

To prove that $\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu = -2\gamma^\sigma\gamma^\lambda\gamma^\nu$.

$$\begin{aligned}
\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu &= \gamma_\mu\gamma^\nu\gamma^\lambda(2g^{\sigma\mu} - \gamma^\mu\gamma^\sigma) \\
&= 2\gamma^\sigma\gamma^\nu\gamma^\lambda - \gamma_\mu\gamma^\nu(\gamma^\mu\gamma^\lambda - 2g^{\mu\lambda})\gamma^\sigma \\
&= 2\gamma^\sigma\gamma^\nu\gamma^\lambda + 2\gamma^\lambda\gamma^\nu\gamma^\sigma + \gamma_\mu(-\gamma^\mu\gamma^\nu + 2g^{\mu\nu})\gamma^\lambda\gamma^\sigma \\
&= 2\gamma^\sigma\gamma^\nu\gamma^\lambda + 2\gamma^\lambda\gamma^\nu\gamma^\sigma - 4\gamma^\nu\gamma^\lambda\gamma^\sigma + 2\gamma^\nu\gamma^\lambda\gamma^\sigma \\
&= 2\gamma^\sigma\gamma^\nu\gamma^\lambda - 2\gamma^\sigma g^{\lambda\nu} \\
&= 2\gamma^\sigma(\gamma^\nu\gamma^\lambda - g^{\lambda\nu}) \\
&= -2\gamma^\sigma\gamma^\lambda\gamma^\nu
\end{aligned}$$

Question 3 (a) (i)

$\left|\frac{3}{2}\ \frac{3}{2}\right\rangle$ can only come from $|1\ 1\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle$

There are hence only two decay modes:

$(a)\ \Delta^{++} \rightarrow \Sigma^+ + K^+$

$(b)\ \Delta^{++} \rightarrow p + \pi^+$

Question 3 (a) (ii)

$\Delta^- = \left|\frac{3}{2}, -\frac{3}{2}\right\rangle = |1, -1\rangle\left|\frac{1}{2}, -\frac{1}{2}\right\rangle$

The 2 decay modes:

$\Delta^- \rightarrow \pi^- + n$

$\Delta^- \rightarrow \Sigma^- + K^0$

Question 3 (b)

$$|1\,0\rangle|1\,0\rangle = \sqrt{\tfrac{2}{3}}\,|2\,0\rangle - \sqrt{\tfrac{1}{3}}\,|0\,0\rangle = \pi^0\pi^0$$

$$|1\,1\rangle|1\,{-1}\rangle = \sqrt{\tfrac{1}{6}}\,|2\,0\rangle + \sqrt{\tfrac{1}{2}}\,|1\,0\rangle + \sqrt{\tfrac{1}{3}}\,|0\,0\rangle$$

$$\text{Rate}\left(\frac{f \to \pi^0\pi^0}{f \to \pi^+\pi^-}\right) = \left(\sqrt{\tfrac{1}{3}}\right)^2 \Big/ \left(\sqrt{\tfrac{1}{3}}\right)^2 = 1$$

Question 3 (c)

The scattering cross section for e^+e^- into hadrons is $\sigma = \frac{\pi}{3}\left(\frac{\hbar Q c\alpha}{E}\right)^2$, compared to the scattering of e^+e^- to $\mu^+\mu^-$: the ratio of the scattering rate is $\frac{2}{3}$, when the existence of colour is taken into account, the theoretical rate is multiplied by a factor of 3 and the resulting rate is 2 which matches experimental results.

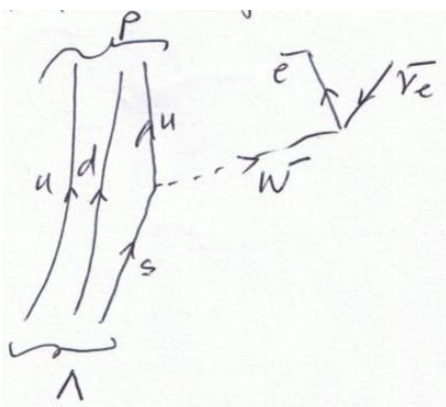
The wavefunction of Δ^{++} is symmetrical but since Δ^{++} is a fermi, it's wave function has to be antisymmetric. With the inclusion of colours wavefunction the Δ^{++} wavefunction becomes anti-symmetric.

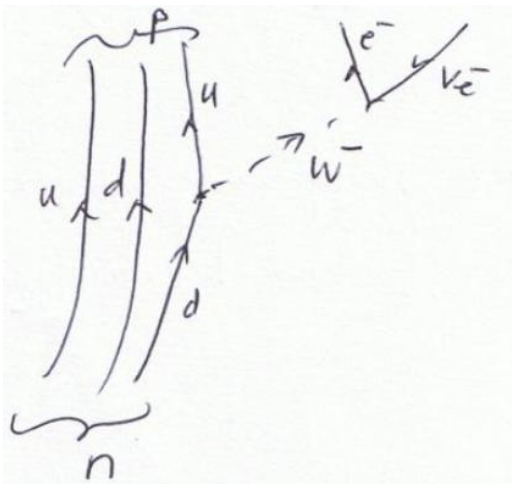
Question 3 (d)

The intrinsic parity of neutrino is -1. Neutrino is an eigenstates of parity.

Question 3 (e)

$(i)\ \Lambda(uds) \to p^+ + e^- + \bar{\nu}_e$ (ii) $n \to p^+ + e^- + \bar{\nu}_e$



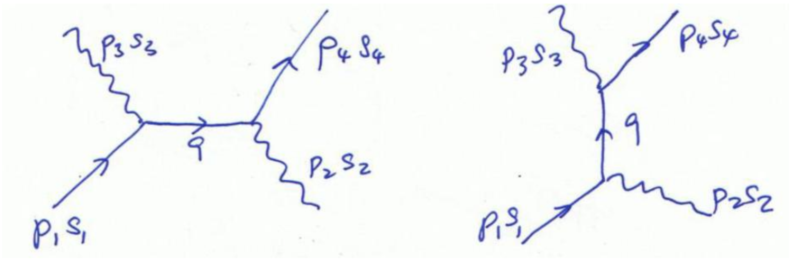


$Rate(i) \propto \sin^2\theta_c\,,\qquad Rate(ii) \propto \cos^2\theta_c$

$$\frac{Rate(i)}{Rate(ii)} \approx \tan^2\theta_c = \frac{1}{20}$$

Question 4 (a)

$e^- + \gamma \to \gamma + e^-$



Question 4 (b)

Applying Feynman rules to 1st diagram,

$$(2\pi)^4 \int \epsilon_\mu(2)\left[\bar{u}(4)(ig_e\gamma^\mu)\frac{i(\not{q}+mc)}{q^2-m^2c^2}(ig_e\gamma^\nu)\,u(1)\right]\epsilon_\nu^*(3)\delta^4(p_1-p_3-q)\delta^4(p_2+q-p_4)\ d^4q$$

$$= -(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\frac{ig_e^2}{(p_1-p_3)^2-m^2c^2}\epsilon_\mu(2)\bar{u}(4)\gamma^\mu(\not{p}_1-\not{p}_3+mc)\,\gamma^\nu u(1)\epsilon_\nu^*(3)$$

$$M_1 = \frac{g_e^2}{(p_1-p_3)^2-m^2c^2}\epsilon_\mu(2)\bar{u}(4)\gamma^\mu(\not{p}_1-\not{p}_3+mc)\,\gamma^\nu u(1)\epsilon_\nu^*(3)$$

Similarly,

$$M_2 = \frac{g_e^2}{(p_1+p_2)^2-m^2c^2}\bar{u}(4)\not{\epsilon}_\mu^*(3)(\not{p}_1+\not{p}_2+mc)\not{\epsilon}_\nu(2)\,u(1)$$

$$M = M_1 + M_2$$

Question 4 (c)

$$\langle|M_1|^2\rangle = \frac{1}{4}\sum_{s_1s_2s_3s_4}\left[\frac{g_e^2}{(p_1-p_3)^2-m^2c^2}\right]^2[\bar{u}(4)\not{\epsilon}(2)(\not{p}_1-\not{p}_3+mc)\not{\epsilon}^*(3)\,u(1)]\,[\bar{u}(4)\not{\epsilon}(2)(\not{p}_1-\not{p}_3+mc)\not{\epsilon}^*(3)\,u(1)]^*$$

Now,

$$\left[u^\dagger(4)\gamma^0\gamma^\mu\epsilon_\mu(2)(\not{p}_1-\not{p}_3+mc)\gamma^\nu\epsilon_\nu^*(3)\,u(1)\right]^\dagger$$

$$= u^\dagger(1)\epsilon_\nu(3)\gamma^{\nu\dagger}\left(\not{p}_1^\dagger-\not{p}_3^\dagger+mc\right)\epsilon_\mu^*(2)\gamma^{\mu\dagger}\gamma^{0\dagger}u(4)$$

$$= \bar{u}(1)\gamma^0\gamma^{\nu\dagger}\epsilon_\nu(3)\gamma^0\gamma^0\left(\not{p}_1^\dagger-\not{p}_3^\dagger+mc\right)\gamma^0\epsilon_\mu^*(2)\gamma^0\gamma^{\mu\dagger}\gamma^{0\dagger}u(4)$$

$$= \bar{u}(1)\not{\epsilon}(3)(\not{p}_1-\not{p}_3+mc)\not{\epsilon}^*(2)u(4)$$

Next do the s_1, s_4 spin summation:

$$\sum_{s_1s_4}\bar{u}(4)\not{\epsilon}(2)(\not{p}_1-\not{p}_3+mc)\not{\epsilon}^*(3)\,u(1)\bar{u}(1)\not{\epsilon}(3)(\not{p}_1-\not{p}_3+mc)\not{\epsilon}^*(2)u(4)$$

$$= Tr[\not{\epsilon}(2)(\not{p}_1-\not{p}_3+mc)\not{\epsilon}^*(3)\,(\not{p}_1+mc)\not{\epsilon}(3)(\not{p}_1-\not{p}_3+mc)\not{\epsilon}^*(2)\,(\not{p}_4+mc)]$$

Now do the s_2, s_3 summation:

$$\sum_{s_2 s_3}(\sim) = \left[\sum_{s_2}\epsilon_\mu(2)\epsilon^*_\lambda(2)\sum_{s_3}\epsilon_\kappa(3)\epsilon^*_\nu(3)\right] Tr\left[\gamma^\mu(\not{p}_1-\not{p}_3+mc)\gamma^\nu\,(\not{p}_1+mc)\gamma^\kappa(\not{p}_1-\not{p}_3+mc)\gamma^\lambda\,(\not{p}_4+mc)\right]$$

$$\therefore \langle|M_1|^2\rangle = \frac{1}{4}\left[\frac{g_e^2}{(p_1-p_3)^2-m^2c^2}\right]^2 Q_{\mu\lambda}Q_{\kappa\nu}Tr\left[\gamma^\mu(\not{p}_1-\not{p}_3+mc)\gamma^\nu\,(\not{p}_1+mc)\gamma^\kappa(\not{p}_1-\not{p}_3+mc)\gamma^\lambda\,(\not{p}_4+mc)\right]$$

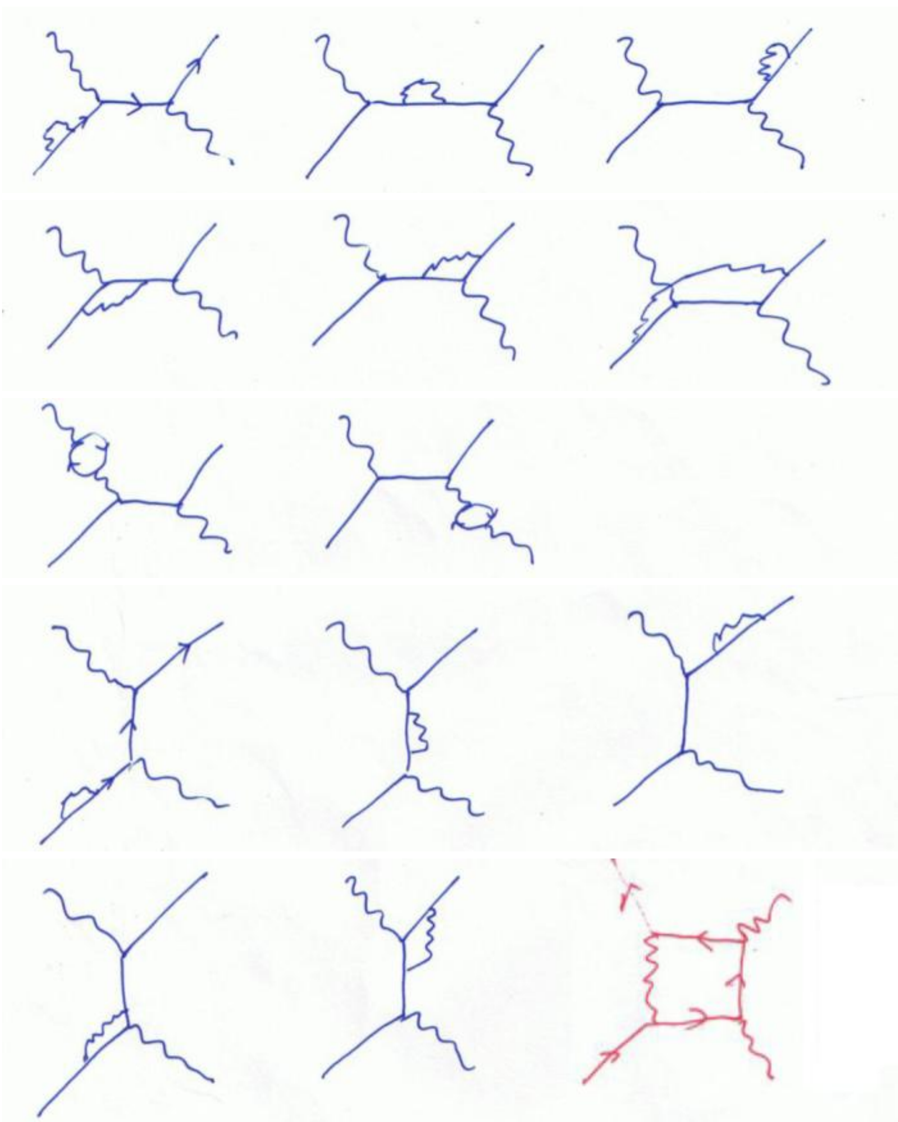
Question 4 (d)

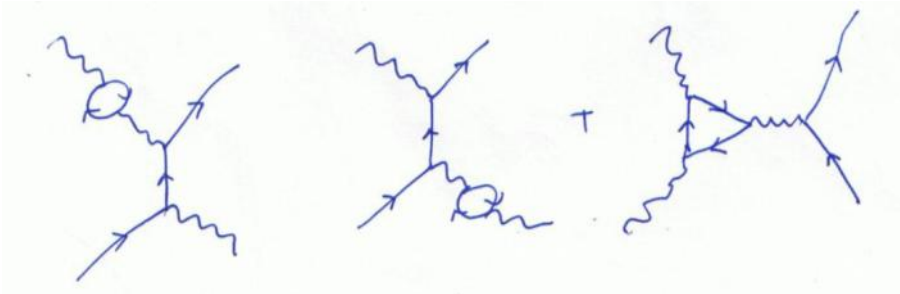
$$\langle M_1 M_2^*\rangle = \frac{1}{4}\sum_{s_1 s_2 s_3 s_4}\frac{g_e^2}{(p_1-p_3)^2-m^2c^2}\frac{g_e^2}{(p_1+p_2)^2-m^2c^2}[\bar{u}(4)\not{\epsilon}(2)(\not{p}_1-\not{p}_3+mc)\not{\epsilon}^*(3)\,u(1)][\bar{u}(1)\not{\epsilon}^*(2)(\not{p}_1+\not{p}_2+mc)\not{\epsilon}(3)\,u(4)]$$

$$\sum_{s_1,s_4}[\,] = Tr[\not{\epsilon}(2)(\not{p}_1-\not{p}_3+mc)\not{\epsilon}^*(3)\,(\not{p}_1+mc)\not{\epsilon}^*(2)(\not{p}_1+\not{p}_2+mc)\not{\epsilon}(3)\,(\not{p}_4+mc)]$$

$$\langle M_1 M_2^*\rangle = \frac{1}{4}\frac{g_e^4}{[(p_1-p_3)^2-m^2c^2][(p_1+p_2)^2-m^2c^2]}Q_{\mu\kappa}Q_{\lambda\nu}Tr\left[\gamma^\mu(\not{p}_1-\not{p}_3+mc)\gamma^\nu\,(\not{p}_1+mc)\gamma^\kappa(\not{p}_1+\not{p}_2+mc)\gamma^\lambda\,(\not{p}_4+mc)\right]$$

Question 4 (e)





<u>Solutions provided by:</u>

Prof. Teo Kien Boon (Questions **1a-b, 2a, 2b(iii), 3a(i), 3e, 4**)

Bong Kok Wei (Questions **1c-d, 2b(i)-(ii), 3a(ii)-d**)