NATIONAL UNIVERSITY OF SINGAPORE

PC4245 PARTICLE PHYSICS

(Semester II: AY 2008-09)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **SIX (6)** printed pages.
2. Answer **ANY THREE (3)** questions.
3. Answers to the questions are to be written in the answer books.
4. This is a **CLOSED BOOK** examination.
5. **One Help Sheet (A4 size, both sides) is allowed** for this examination.
6. The Clebsch-Gordan coefficient table is attached on the last printed page.
7. A Table of Constants will be supplied.

1. (a) Show that the invariant mass of a pair of photons of energies E_1 and E_2 with angle θ between their directions of motion is given by the expression

$$M^2 = \frac{4E_1E_2}{c^4}\sin^2\left(\frac{\theta}{2}\right)$$

(b) Consider the decay of a particle of mass m and energy E into two photons, and show that the minimum opening angle between the photons is given by:

$$\sin\left(\frac{\theta_{min}}{2}\right) = \frac{mc^2}{E}$$

A particle of energy 10 GeV decays to two photons with an opening angle of 2°. Could this particle be an η meson (of mass 549 MeV) or a π^0 meson (of mass 135 MeV)?

(c) Consider the decay $A^0 \rightarrow \gamma + \gamma$.

Given that the amplitude for the process is $M(\vec{p}_2, \vec{p}_3)$, where $\vec{p}_2$ and $\vec{p}_3$ are respectively the 3-momenta of the two outgoing photons,

(i) find the decay rate in terms of m_{A^0} and $M(\vec{p}_2, \vec{p}_3)$, where m_{A^0} is the mass of A^0.

(ii) What are the values of $|\vec{p}_2|$ and $|\vec{p}_3|$?

Note : The following formula for the decay process $1 \rightarrow 2 + 3$ can be used:

$$d\Gamma = \frac{S}{2\hbar m_1}|M|^2 \frac{d^3\vec{p}_2}{(2\pi)^3 2p_2^0}\frac{d^3\vec{p}_3}{(2\pi)^3 2p_3^0}(2\pi)^4\delta^{(4)}(p_1 - p_2 - p_3)$$

(d) Consider the annihilation of positronium (e^+e^-) into gamma photons from the S-state (L=0). Using arguments from charge conjugation invariance and energy-momentum conservation, explain why para-positronium (where e^+ and e^- have opposite spins) is likely to decay to two photons while ortho-positronium (where e^+ and e^- have parallel spins) is likely to decay to three photons.

2. (a) The Dirac Hamiltonian is given by:

$$H = c\left[\vec{\alpha}\cdot\vec{p} + \beta\,\mathrm{mc}\right] \quad .$$

Using this Hamiltonian, construct the normalized spinors $u^{(+)}$ and $u^{(-)}$ representing an electron of momentum $\vec{p}$ with helicity ± 1 . That is, find the u*'s* that satisfy the Dirac Hamiltonian equation with positive energy E, and are at the same time eigenspinors of the helicity operator $\dfrac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}$ with eigenvalues ± 1.

Hints:

- Use the following normalization condition:

$$u^{+}u = 2\mathrm{E}/\mathrm{c}$$

- $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix}$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(b) (i) Simplify the following expression:

$$\gamma^{\mu}\gamma^{\nu}\left(1-\gamma^{5}\right)\gamma^{\lambda}\left(1+\gamma^{5}\right)\gamma_{\lambda}$$

(ii) Prove that the trace of the product of an odd number of gamma matrices is zero.

(iii) Prove the following trace theorem:

$$\gamma_{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\sigma}\gamma^{\mu} = -2\gamma^{\sigma}\gamma^{\lambda}\gamma^{\nu}$$

Hint - The following formulae may be used without proof:

$$\left\{\gamma^{\mu},\gamma^{\nu}\right\} = 2g^{\mu\nu} \quad ; \quad \left\{\gamma^{5},\gamma^{\mu}\right\} = 0$$

3. (a) Members of the baryon decuplet typically decay after 10^{-23} seconds into a lighter baryon (from the baryon octet) and a meson (from the pseudoscalar meson octet). Using isospin arguments, deduce all the possible strong decay modes for

(i) Δ^{++} (ii) Δ^{-}

Hints:

- Members of the baryon octet form the following isospin multiplets:

$$\begin{pmatrix} p \\ n \end{pmatrix}, \begin{pmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{pmatrix}, \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix}, \Lambda$$

- Members of the pseudoscalar meson octet form the following isospin multiplets:

$$\begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}, \begin{pmatrix} \overline{K}^0 \\ K^- \end{pmatrix}, \eta$$

(b) Deduce the ratio of the decay modes of the f meson (I=0) :

$$\frac{f \to \pi^0\pi^0}{f \to \pi^+\pi^-} \quad .$$

(Note: the Clebsch-Gordan coefficient table is attached.)

(c) Discuss two pieces of experimental evidence for the existence of colour in QCD.

(d) Is the neutrino an eigenstate of P? If so, what is its intrinsic parity?

(e) Draw the Feynman diagrams for the following weak decays:

(i) $\Lambda(uds) \to p^+ + e^- + \bar{\nu}_e$

(ii) $n \to p^+ + e^- + \bar{\nu}_e$

Use Cabibbo theory to comment on their relative decay rates.

4. (a) Draw the lowest-order Feynman diagrams for Compton scattering:

$$e^- + \gamma \rightarrow \gamma + e^-$$

(b) Using the Feynman rules for quantum electrodynamics, obtain the scattering amplitude $M = M_1 + M_2$ for the above process.

(c) Prove that the spin-averaged amplitude $\langle|M_1|^2\rangle$ for Compton scattering, averaged over both electron and photon spins, is given by:

$$\langle |M_1|^2 \rangle = \left[\frac{g_e^2/2}{(p_1 - p_3)^2 - m^2c^2} \right]^2 Q_{\mu\lambda}\, Q_{\kappa\nu}$$
$$\times \ \mathrm{Tr}\left[\gamma^\mu (\not{p}_1 - \not{p}_3 + mc)\gamma^\nu (\not{p}_1 + mc)\gamma^\kappa (\not{p}_1 - \not{p}_3 + mc)\gamma^\lambda (\not{p}_4 + mc) \right]$$

where $$Q_{\mu\nu} \equiv \begin{cases} 0 & \text{if } \mu \text{ or } \nu \text{ is } 0 \\ \delta_{ij} - \hat{p}_i \hat{p}_j & \text{otherwise} \end{cases}$$

(d) Derive the corresponding expression for the spin-averaged $\langle M_1 M_2^* \rangle$.

(e) Draw all 17 fourth-order (four-vertex) Feynman diagrams for Compton scattering.

Note - The following formulae can be used without proof:

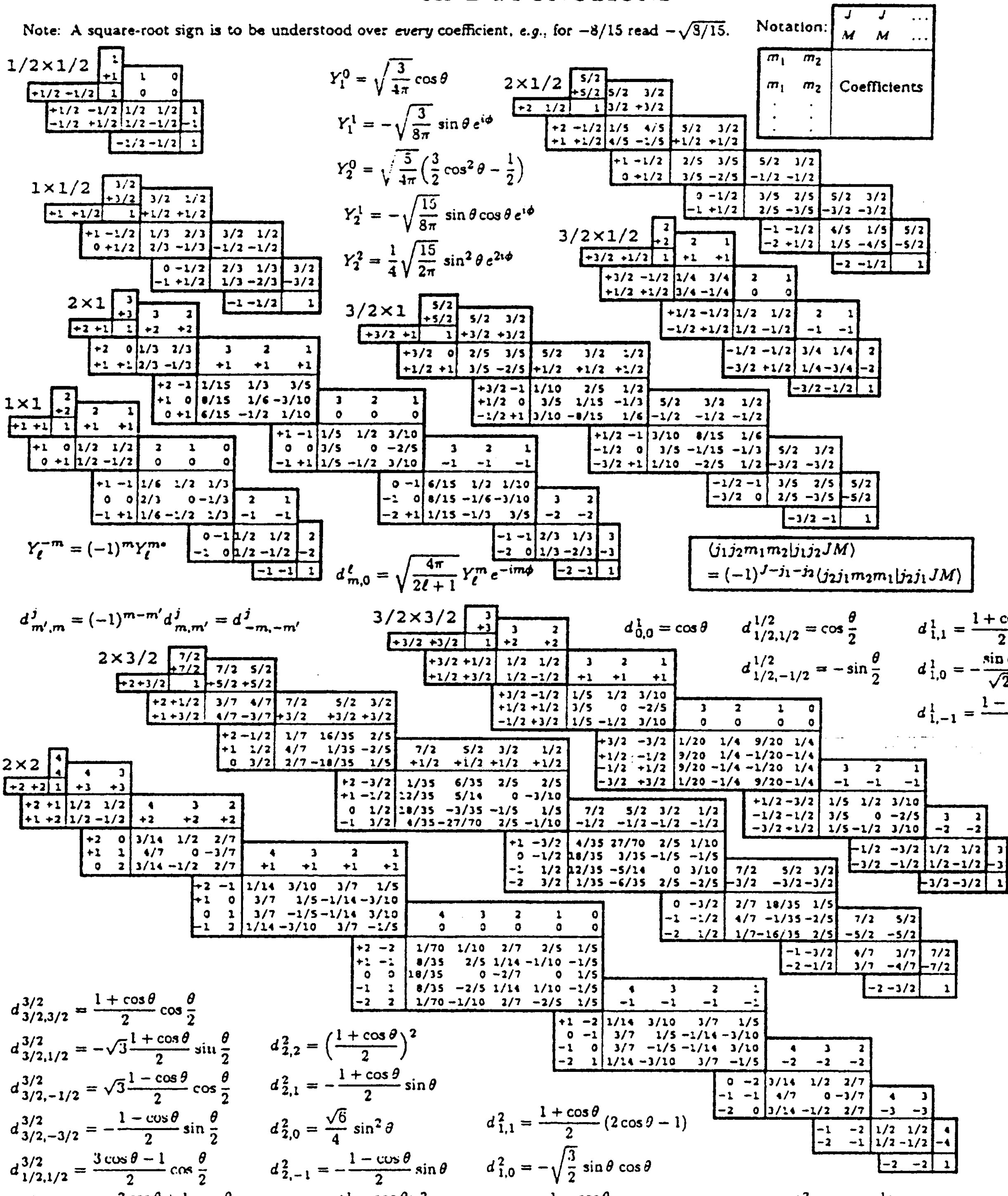
- $\sum_s u^{(s)}(p)\, \bar{u}^{(s)}(p) = \not{p} + mc$
- $\sum_s v^{(s)}(p)\, \bar{v}^{(s)}(p) = \not{p} - mc$
- $\gamma^0 \gamma^{\mu +} \gamma^0 = \gamma^\mu$
- $\sum_s \varepsilon_\mu^{(s)}\, \varepsilon_\nu^{*(s)} = Q_{\mu\nu}$

(TKB)

- END OF PAPER -

32. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.



$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1)$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta - 1}{2}\cos\frac{\theta}{2} \qquad d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta + 1}{2}\sin\frac{\theta}{2} \qquad d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2 \qquad d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta + 1) \qquad d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 32.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.