NATIONAL UNIVERSITY OF SINGAPORE

PC4245 PARTICLE PHYSICS

(Semester II: AY 2009-10)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **SIX (6)** printed pages.
2. Answer **ANY THREE (3)** questions.
3. Answers to the questions are to be written in the answer books.
4. This is a **CLOSED BOOK** examination.
5. **One Help Sheet (A4 size, both sides) is allowed** for this examination.
6. The Clebsch-Gordan coefficient table is attached as the last printed page.
7. A Table of Constants will be supplied.

Question 1

(a) Consider the pair annihilation process $e^+ + e^- \rightarrow \gamma + \gamma$ in the CM frame.

(i) Show that the differential cross section, in the usual notations, can be written as

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{16\pi}\right)^2 \frac{|\mathrm{M}|^2}{2E\sqrt{E^2 - m^2c^4}}$$

Here E and m are the incident energy and mass of the electron respectively.

(ii) What is the value of the momentum of either outgoing photon?

Note that the following formula can be used:

$$d\sigma = |\mathrm{M}|^2 \frac{\hbar^2 S}{4}\left[(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2\right]^{-\frac{1}{2}} \frac{d^3\vec{p}_3}{(2\pi)^3 2p_3^0} \frac{d^3\vec{p}_4}{(2\pi)^3 2p_4^0} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)$$

(b) Consider the following electron-positron annihilation process into three photons:

$$e^+ + e^- \rightarrow \gamma + \gamma + \gamma$$

Suppose we look for events in the CM frame in which the three final photons have the same energy. Prove in this case that the three photons emerge at an angle of 120^0 with respect to one another.

(c) Explain the differences between invariants and conserved quantities in any typical scattering event.

Question 2

(a) Consider all proton-pion collisions that result in a final state consisting of an isotriplet Σ baryon and an isodoublet kaon i.e. $p\,\pi \rightarrow \Sigma\, K$.

(i) Show that there are only five such processes compatible with isospin conservation.

(ii) Relate the ratios of the cross sections of these processes in the case when the I=3/2 channel dominates.

Hints: Nucleons, pions, sigmas and kaons form the following isospin multiplets:

$$\begin{pmatrix} p \\ n \end{pmatrix}, \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}, \begin{pmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{pmatrix}, \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \begin{pmatrix} \overline{K}^0 \\ K^- \end{pmatrix}.$$

(b) The deuteron is a bound state of one proton and one neutron.

(i) Deduce its isospin.

(ii) Using Clebsch-Gordan coefficients, evaluate the following ratio:

$$\frac{\text{Rate}\left(n+n \rightarrow d+\pi^-\right)}{\text{Rate}\left(p+n \rightarrow d+\pi^0\right)}$$

(c) The η (545MeV) is a pseudoscalar meson ($J^P = 0^-$), just like the pions.

Use parity arguments to explain why the η cannot have a strong decay into $\pi^+\pi^-$, but can decay into $\pi^+\pi^-\pi^0$.

Question 3

(a) Draw the lowest order Feynman diagram for the following decays:

$$\Sigma^-(1197\text{ MeV, dds}) \rightarrow n + e^- + \bar{\nu}_e$$

$$\Sigma^+(1189\text{ MeV, uus}) \rightarrow n + e^+ + \nu_e$$

Hence, or otherwise, explain the smallness of the ratio of their decay widths:

$$\frac{\Gamma\left(\Sigma^+ \rightarrow n + e^+ + \nu_e\right)}{\Gamma\left(\Sigma^- \rightarrow n + e^- + \bar{\nu}_e\right)} < 5 \times 10^{-3} \quad .$$

(b) Use Cabibbo theory to explain the experimental observation that the decay of a charmed hadron almost always leads to a strange hadron in the final state.

(c) Starting from the Dirac equation given by:

$$\left(i\hbar\gamma^\mu \partial_\mu - mc\right)\psi = 0$$

derive the Dirac Hamiltonian H:

$$H\psi = \left(c\vec{\alpha}\cdot\vec{p} + \beta mc^2\right)\psi$$

Note:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Question 4

(a) Draw the lowest-order Feynman diagram for electron-muon scattering:

$$e^- + \mu^- \rightarrow e^- + \mu^-$$

Using the Feynman rules for quantum electrodynamics, obtain the scattering amplitude M for the above process.

Using the Casimir trick, show that the spin-averaged quantity $<|M|^2>$ is given by:

$$\langle |M|^2 \rangle = \frac{8g_e^4}{(p_1 - p_3)^4}$$
$$\times \left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)(m_\mu c)^2 - (p_2 \cdot p_4)(m_e c)^2 + 2\left(m_e m_\mu c^2\right)^2 \right]$$

Note: The following formulae can be used without proof:

- $\sum_s u^{(s)}(p)\, \bar{u}^{(s)}(p) = \not{p} + mc$
- $\sum_s v^{(s)}(p)\, \bar{v}^{(s)}(p) = \not{p} - mc$
- $Tr\left[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma\right] = 4\left[g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda}\right]$
- $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\, g^{\mu\nu}$; $\gamma^0 \gamma^{\nu+} \gamma^0 = \gamma^\nu$

(b) Draw any 6 fourth-order (four-vertex) Feynman diagrams for electron-muon scattering.

(c) Draw the two lowest-order Feynman diagrams for pair annihilation:

$$e^- + e^+ \rightarrow \gamma + \gamma$$

Using the Feynman rules for quantum electrodynamics, obtain the scattering amplitude $M = M_1 + M_2$ for the above process.

(TKB)

- END OF PAPER -

32. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2			
m_1 m_2	Coefficients		
. .			

1/2×1/2

m_1 m_2	J=1, M=+1
+1/2 +1/2	1

m_1 m_2	J=1, M=0	J=0, M=0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	J=1, M=−1
−1/2 −1/2	1

1×1/2

m_1 m_2	J=3/2, M=+3/2
+1 +1/2	1

m_1 m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	J=3/2, M=−3/2
−1 −1/2	1

2×1/2

m_1 m_2	J=5/2, M=+5/2
+2 +1/2	1

m_1 m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	J=5/2, M=−5/2
−2 −1/2	1

3/2×1/2

m_1 m_2	J=2, M=+2
+3/2 +1/2	1

m_1 m_2	J=2, M=+1	J=1, M=+1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	J=2, M=0	J=1, M=0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	J=2, M=−1	J=1, M=−1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	J=2, M=−2
−3/2 −1/2	1

2×1

m_1 m_2	J=3, M=+3
+2 +1	1

m_1 m_2	J=3, M=+2	J=2, M=+2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	6/15	−1/2	1/10

m_1 m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0 −1	6/15	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	J=3, M=−2	J=2, M=−2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	J=3, M=−3
−2 −1	1

1×1

m_1 m_2	J=2, M=+2
+1 +1	1

m_1 m_2	J=2, M=+1	J=1, M=+1
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	J=2, M=−1	J=1, M=−1
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	J=2, M=−2
−1 −1	1

3/2×1

m_1 m_2	J=5/2, M=+5/2
+3/2 +1	1

m_1 m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	J=5/2, M=−5/2
−3/2 −1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

2×3/2

m_1 m_2	J=7/2, M=+7/2
+2 +3/2	1

m_1 m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m_1 m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

m_1 m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

m_1 m_2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

m_1 m_2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

m_1 m_2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m_1 m_2	J=7/2, M=−7/2
−2 −3/2	1

3/2×3/2

m_1 m_2	J=3, M=+3
+3/2 +3/2	1

m_1 m_2	J=3, M=+2	J=2, M=+2
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m_1 m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m_1 m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m_1 m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m_1 m_2	J=3, M=−2	J=2, M=−2
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m_1 m_2	J=3, M=−3
−3/2 −3/2	1

2×2

m_1 m_2	J=4, M=+4
+2 +2	1

m_1 m_2	J=4, M=+3	J=3, M=+3
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m_1 m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

m_1 m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

m_1 m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

m_1 m_2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

m_1 m_2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m_1 m_2	J=4, M=−3	J=3, M=−3
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m_1 m_2	J=4, M=−4
−2 −2	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1)$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta - 1}{2}\cos\frac{\theta}{2} \qquad d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta + 1}{2}\sin\frac{\theta}{2} \qquad d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2 \qquad d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta + 1) \qquad d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 32.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.