NATIONAL UNIVERSITY OF SINGAPORE

PC4245 PARTICLE PHYSICS

(Semester II: AY 2010 -11)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **SIX (6)** printed pages.
2. Answer **ANY THREE (3)** questions.
3. Answers to the questions are to be written in the answer books.
4. This is a **CLOSED BOOK** examination.
5. **One Help Sheet (A4 size, both sides) is allowed** for this examination.
6. The Clebsch-Gordan coefficient table is attached as the last printed page.

Question 1

(a) A particle of mass m_1 decays into two secondaries of masses m_2 and m_3 .

(i) If the amplitude for the process is $M(\vec{p}_2, \vec{p}_3)$, find the decay rate.

(ii) What are the values of the two outgoing momenta $|\vec{p}_2|$ and $|\vec{p}_3|$, in terms of the three particle masses?

Note : The formula for the decay process $1 \rightarrow 2 + 3$ can be used:

$$d\Gamma = \frac{S}{2\hbar m_1} |\mathrm{M}|^2 \frac{d^3\vec{p}_2}{(2\pi)^3 2p_2^0} \frac{d^3\vec{p}_3}{(2\pi)^3 2p_3^0} (2\pi)^4 \delta^{(4)}(p_1 - p_2 - p_3)$$

(b) Briefly describe two different ways in which one can look for experimental evidence for time reversal violation in particle physics.

Question 2

(a) Explain, with the help of diagrams (if necessary), the very small branching ratios ($B\Re$) of the following decays i.e. why the decays are forbidden or heavily suppressed:

(i) $B\Re\left(K^+ (u\bar{s}) \rightarrow \pi^+ \nu \bar{\nu}\right) = 1.73 \times 10^{-10}$

(ii) $\dfrac{B\Re(\pi^0 \rightarrow \gamma\gamma\gamma)}{B\Re(\pi^0 \rightarrow \gamma\gamma)} < 4 \times 10^{-7}$.

(iii) $B\Re\left(D^0 (c\bar{u}) \rightarrow \mu^+ \mu^-\right) < 5.3 \times 10^{-7}$

(iv) $\eta\ (J^P = 0^-) \nrightarrow \pi^+ \pi^-$

(b) Use the OZI rule to explain why, unlike the ω meson, the dominant decay mode of the ϕ meson is into 2 kaons and not into 3 pions.

$$|\phi\rangle = s\bar{s} \quad ; \quad |\omega\rangle = \frac{1}{\sqrt{2}}\left(u\bar{u} + d\bar{d}\right)$$

Question 3

(a) The scattering amplitude M for muon decay:

$$\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e$$

is given by:

$$M = \frac{g_W^2}{8(M_W c)^2}\left[\bar{u}^{(s_3)}(p_3)\gamma^\mu(1-\gamma^5)u^{(s_1)}(p_1)\right]\left[\bar{u}^{(s_4)}(p_4)\gamma_\mu(1-\gamma^5)u^{(s_2)}(p_2)\right]$$

Using Casimir's trick and the appropriate trace theorems, prove that the spin-averaged amplitude $\langle|M|^2\rangle$ is given by

$$\langle|M|^2\rangle = 2\left(\frac{g_W}{M_W c}\right)^4 (p_1 \cdot p_2)(p_3 \cdot p_4)$$

Note - The following formulae can be used without proof:

- $\sum_s u^{(s)}(p)\,\bar{u}^{(s)}(p) = \not{p} + mc$
- $\gamma^0 \gamma^{\mu+} \gamma^0 = \gamma^\mu$
- $Tr\left[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma\right] = 4\left[g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}\right]$
- $\mathrm{Tr}\left(\gamma^5 \gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\right) = 4i\varepsilon^{\mu\nu\lambda\sigma}$

  where

$$\varepsilon^{\mu\nu\lambda\sigma} \equiv \begin{cases} -1, & \text{if } \mu\nu\lambda\sigma \text{ is an even permutation of } 0123, \\ +1, & \text{if } \mu\nu\lambda\sigma \text{ is an odd permutation of } 0123, \\ 0, & \text{if any two indices are the same.} \end{cases}$$

- $\varepsilon^{\mu\nu\lambda\sigma}\varepsilon_{\mu\nu\kappa\tau} = -2\left(\delta^\lambda_\kappa \delta^\sigma_\tau - \delta^\lambda_\tau \delta^\sigma_\kappa\right)$

(b) Draw the two lowest-order Feynman diagrams for electron-positron scattering:

$$e^- + e^+ \rightarrow e^- + e^+$$

Using the Feynman rules for quantum electrodynamics, obtain the scattering amplitude M for the above process.

Question 4

(a) All Δ particles (Δ^{++}, Δ^{+}, Δ^{0}, Δ^{-}) decay quickly to a combination of a pion and nucleon. Show that there are six such decays that conserve charge, and calculate the relative ratios of their decay rates.

(b) In general, non-leptonic decays of strange particles are characterized by the rule ΔS=1, ΔI=½ , which arises from the exchange of a strange quark s with a non-strange quark d.
Use this rule to compute the ratio of decay rates for:

(i) $K_S \to \pi^+\pi^- \; / \; K_S \to \pi^0\pi^0$

(ii) $\Xi^- \to \Lambda\pi^- \; / \; \Xi^0 \to \Lambda\pi^0$

[Hint: The ΔI=½ rule may be applied by introducing a hypothetical particle (sometimes called a "spurion") of I=½ to the left hand side of the reaction, and then treating the decay as an isospin-conserving reaction.

The particles involved are grouped in isospin multiplets: $\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$, $\begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix}$, Λ]

(c) Suppose that u(p) is a bispinor satisfying the momentum-space Dirac equation as follows:

$$\left(\gamma^{\mu} p_{\mu} - \mathrm{mc}\right) u(p) = 0 .$$

(i) Derive the following:

$$\gamma^5 u(p) = \begin{pmatrix} \dfrac{c(\vec{p}.\vec{\sigma})}{E+mc^2} & 0 \\ 0 & \dfrac{c(\vec{p}.\vec{\sigma})}{E-mc^2} \end{pmatrix} u(p)$$

(ii) If the particle in question is massless, express $\gamma^5 u(p)$ in terms of the helicity operator $\hat{p}.\vec{\Sigma}$.

(iii) Hence, or otherwise, construct a bispinor $u^{(+)}$ representing a massless Dirac fermion with helicity +1.

Note: Use the following representation for gamma matrices:

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

(TKB)

- END OF PAPER -

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND *d* FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2			
m_1 m_2	Coefficients		
⋮ ⋮			

1/2×1/2

m1 m2	1 +1
+1/2 +1/2	1

m1 m2	1 0	0 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m1 m2	1 −1
−1/2 −1/2	1

1×1/2

m1 m2	3/2 +3/2
+1 +1/2	1

m1 m2	3/2 +1/2	1/2 +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m1 m2	3/2 −1/2	1/2 −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m1 m2	3/2 −3/2
−1 −1/2	1

2×1

m1 m2	3 +3
+2 +1	1

m1 m2	3 +2	2 +2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m1 m2	3 +1	2 +1	1 +1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

m1 m2	3 0	2 0	1 0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m1 m2	3 −1	2 −1	1 −1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m1 m2	3 −2	2 −2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m1 m2	3 −3
−2 −1	1

1×1

m1 m2	2 +2
+1 +1	1

m1 m2	2 +1	1 +1
+1 0	1/2	1/2
0 +1	1/2	−1/2

m1 m2	2 0	1 0	0 0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m1 m2	2 −1	1 −1
0 −1	1/2	1/2
−1 0	1/2	−1/2

m1 m2	2 −2
−1 −1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\,Y_\ell^m\, e^{-im\phi}$$

2×1/2

m1 m2	5/2 +5/2
+2 +1/2	1

m1 m2	5/2 +3/2	3/2 +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m1 m2	5/2 +1/2	3/2 +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m1 m2	5/2 −1/2	3/2 −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m1 m2	5/2 −3/2	3/2 −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m1 m2	5/2 −5/2
−2 −1/2	1

3/2×1/2

m1 m2	2 +2
+3/2 +1/2	1

m1 m2	2 +1	1 +1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m1 m2	2 0	1 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m1 m2	2 −1	1 −1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m1 m2	2 −2
−3/2 −1/2	1

3/2×1

m1 m2	5/2 +5/2
+3/2 +1	1

m1 m2	5/2 +3/2	3/2 +3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m1 m2	5/2 +1/2	3/2 +1/2	1/2 +1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m1 m2	5/2 −1/2	3/2 −1/2	1/2 −1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m1 m2	5/2 −3/2	3/2 −3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m1 m2	5/2 −5/2
−3/2 −1	1

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

2×3/2

m1 m2	7/2 +7/2
+2 +3/2	1

m1 m2	7/2 +5/2	5/2 +5/2
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m1 m2	7/2 +3/2	5/2 +3/2	3/2 +3/2
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

m1 m2	7/2 +1/2	5/2 +1/2	3/2 +1/2	1/2 +1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

m1 m2	7/2 −1/2	5/2 −1/2	3/2 −1/2	1/2 −1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

m1 m2	7/2 −3/2	5/2 −3/2	3/2 −3/2
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

m1 m2	7/2 −5/2	5/2 −5/2
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m1 m2	7/2 −7/2
−2 −3/2	1

3/2×3/2

m1 m2	3 +3
+3/2 +3/2	1

m1 m2	3 +2	2 +2
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m1 m2	3 +1	2 +1	1 +1
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m1 m2	3 0	2 0	1 0	0 0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m1 m2	3 −1	2 −1	1 −1
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m1 m2	3 −2	2 −2
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m1 m2	3 −3
−3/2 −3/2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1}_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×2

m1 m2	4 +4
+2 +2	1

m1 m2	4 +3	3 +3
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m1 m2	4 +2	3 +2	2 +2
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

m1 m2	4 +1	3 +1	2 +1	1 +1
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

m1 m2	4 0	3 0	2 0	1 0	0 0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

m1 m2	4 −1	3 −1	2 −1	1 −1
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

m1 m2	4 −2	3 −2	2 −2
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m1 m2	4 −3	3 −3
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m1 m2	4 −4
−2 −2	1

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory,* Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra,* Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum,* Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients,* North American Rockwell Science Center, Thousand Oaks, Calif., 1974).