NATIONAL UNIVERSITY OF SINGAPORE

PC4248 RELATIVITY

(Semester I: AY 2013–14)

Examiner: Assoc Prof Edward Teo

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Write your matriculation number only. **Do not write your name.**

2. This examination paper contains **FOUR** questions and comprises **THREE** printed pages.

3. Answer **ALL** questions.

4. Please start each question on a new page.

5. This is a **closed book** examination.

6. Students are allowed to bring in one A4-sized double-sided help sheet.

1. Let $\mathcal{S}$ and $\mathcal{S}'$ be inertial frames of reference, with $\mathcal{S}'$ obtained from $\mathcal{S}$ by a boost in the $+x$-direction with speed v $(0 < v < 1)$. Suppose there is a light source which is at rest in $\mathcal{S}'$, emitting photons isotropically in all directions.

(a) Consider a photon in $\mathcal{S}'$ with energy E', moving in the (x', y')-plane at an angle θ' with respect to the $+x'$-axis. Show that in $\mathcal{S}$, the corresponding energy and angle of the photon are given by

$$E = \frac{E'(1 + v\cos\theta')}{\sqrt{1 - v^2}}, \qquad \cos\theta = \frac{\cos\theta' + v}{1 + v\cos\theta'}.$$

(b) For a photon moving in the $+x'$-direction, calculate its frequency as seen in $\mathcal{S}$ in terms of its original frequency in $\mathcal{S}'$. Does this correspond to a redshift or a blueshift? [*Hint: Recall Planck's equation $E = h\nu$ for photons.*]

(c) Now assume v is close to one. For the photons that are emitted in the forward direction, i.e., $\theta' < \frac{\pi}{2}$, show that they will appear in $\mathcal{S}$ to be concentrated in a narrow cone about $\theta = 0$. Analyse and describe what happens to the photons which are emitted in the *backward* direction, i.e., $\frac{\pi}{2} < \theta' \leq \pi$.

2. (a) Verify directly that if $\Gamma^a_{\ bc} = \frac{1}{2}g^{ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc})$, then $\nabla_a g_{bc} = 0$. Thus show that $\nabla_a g^{bc} = 0$.

(b) Show that if an arbitrary tensor $T_{bc\cdots}$ satisfies $\nabla_a T_{bc\cdots} = 0$, then $\nabla_a T^{bc\cdots} = 0$.

(c) Explain why the condition $R_{ab} = 0$ for a Ricci-flat spacetime does not imply that $R_{abcd} = 0$. Show, however, that $\nabla^a R_{abcd} = 0$ for such a spacetime.

3. Consider the three-dimensional spacetime with metric:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + [\rho(t)]^2\,\mathrm{d}\varphi^2.$$

This describes a spacelike cylinder whose radius $\rho(t)$ varies with time.

(Question continued on next page)

(a) Calculate the Christoffel symbols Γ^a_{bc}.

(b) Calculate the non-zero components of the Riemann tensor $R_{abc}{}^d = \partial_b \Gamma^d_{ac} - \partial_a \Gamma^d_{bc} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{bc}\Gamma^d_{ae}$.

(c) Find the condition on $\rho(t)$ such that the geometry is flat, and solve for $\rho(t)$ explicitly. How many independent continuous symmetries does the spacetime possess in this case?

4. The Schwarzschild metric describing a black hole is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\,.$$

(a) Setting $u = \frac{2m}{r}$, show that null geodesics in the equatorial plane $\theta = \frac{\pi}{2}$ satisfy the equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = \frac{3}{2}u^2. \tag{1}$$

(b) Verify that

$$u = \frac{2}{3}$$

is a solution to (1). Describe the significance of this orbit.

(c) Verify that

$$u = \frac{2}{3} - \frac{2}{1 + \cosh\varphi}$$

is also a solution to (1). Sketch this orbit in (r, φ) coordinates and describe its relation to the orbit in part (b).

[*Hint: Recall* $\sinh x = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})$, $\cosh x = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$, $\cosh^2 x - \sinh^2 x = 1$.]

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