PC4248 Relativity

2013/2014 Examination Model Answers

1. (a) 4-momentum of photon:

$$P^a = (E, p\cos\theta, p\sin\theta, 0)$$

But $P^a P_a = 0$ implies $p = E$, so

$$P^a = E(1, \cos\theta, \sin\theta, 0)$$

with a similar expression in the $\mathcal{S}'$ frame:

$$P'^a = E'(1, \cos\theta', \sin\theta', 0)$$

The two 4-momenta are related by an "inverse" boost in the x-direction:

$$P^a = A^a_b P'^b,$$

where A^a_b is as given by

$$A^a_b = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

First and second components of this equation give

$$E = \gamma E'(1 + v\cos\theta')$$
$$E\cos\theta = \gamma E'(v + \cos\theta')$$

The ratio of these two equations becomes

$$\cos\theta = \frac{\cos\theta' + v}{1 + v\cos\theta'} \qquad (*)$$

(b) For a photon in the x-direction, $\theta = 0$.

$$E = \frac{E'(1+v)}{\sqrt{1-v^2}} = \frac{E'(1+v)}{\sqrt{(1-v)(1+v)}} = E'\sqrt{\frac{1+v}{1-v}}$$

Since $E = h\nu$,

$$\nu = \nu' \sqrt{\frac{1+v}{1-v}} > \nu'$$

This corresponds to a blueshift.

(c) As $v \to 1$, we have from $(*)$,

$$\cos\theta \to \frac{\cos\theta' + 1}{1 + \cos\theta'} = 1$$

i.e., $\theta \to 0$. When v is close to one, θ will be an angle close to zero. So the photons are concentrated in a narrow cone about $\theta = 0$.

The above limit is clearly not valid when $\theta' = \pi$, since we get $0/0$. We have to evaluate the limit of this case separately:

$$\begin{aligned}\cos\theta &= \frac{-1+v}{1-v} \\ &= -1\end{aligned}$$

Thus, $\theta = \pi$ as well. We conclude that the backward phtons are spead out in the *remaining region outside the narrow cone*.

2. (a)

$$\begin{aligned}
\nabla_a g_{bc} &= \partial_a g_{bc} - \Gamma^d_{ab} g_{dc} - \Gamma^d_{ac} g_{bd} \\
&= \partial_a g_{bc} - \frac{1}{2} g^{de}(\partial_a g_{be} + \partial_b g_{ea} - \partial_e g_{ab}) g_{dc} - \frac{1}{2} g^{de}(\partial_a g_{ce} + \partial_c g_{ea} - \partial_e g_{ac}) g_{bd} \\
&= \partial_a g_{bc} - \frac{1}{2}\delta^e_c(\partial_a g_{be} + \partial_b g_{ea} - \partial_e g_{ab}) - \frac{1}{2}\delta^e_b(\partial_a g_{ce} + \partial_c g_{ea} - \partial_e g_{ac}) \\
&= \partial_a g_{bc} - \frac{1}{2}(\partial_a g_{bc} + \cancel{\partial_b g_{ca}} - \cancel{\partial_c g_{ab}}) - \frac{1}{2}(\partial_a g_{cb} + \cancel{\partial_c g_{ba}} - \cancel{\partial_b g_{ac}}) \\
&= 0
\end{aligned}$$

Take covariant derivative of the equation $g^{bc} g_{cd} = \delta^b_d$.

LHS:

$$\begin{aligned}\nabla_a(g^{bc} g_{cd}) &= (\nabla_a g^{bc}) g_{cd} + g^{bc} \nabla_a g_{cd} \\ &= (\nabla_a g^{bc}) g_{cd}\end{aligned}$$

RHS:

$$\begin{aligned}\nabla_a \delta^b_c &= \partial_a \delta^b_c + \Gamma^b_{ad}\delta^d_c - \Gamma^d_{ac}\delta^b_d \\ &= \Gamma^b_{ac} - \Gamma^b_{ac} \\ &= 0\end{aligned}$$

Hence $\nabla_a g^{bc} = 0$.

(b) By the previous result,

$$\begin{aligned}
\nabla_a T^{bc\cdots} &= \nabla_a (g^{bb'} g^{cc'} \cdots T_{b'c'\cdots}) \\
&= \nabla_a (g^{bb'}) g^{cc'} \cdots T_{b'c'\cdots} + g^{bb'} \nabla_a (g^{cc'}) \cdots T_{b'c'\cdots} + \cdots + g^{bb'} g^{cc'} \cdots \nabla_a T_{b'c'\cdots} \\
&= g^{bb'} g^{cc'} \cdots \nabla_a T_{b'c'\cdots} \\
&= 0
\end{aligned}$$

(c) Since $R_{ab} = R^c{}_{acb}$, its vanishing implies that the sum $R^0{}_{a0b} + R^1{}_{a1b} + R^2{}_{a2b} + R^3{}_{a3b} = 0$. It does not imply that the individual components $R^a{}_{bcd}$ are zero.

From the Bianchi identity

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0$$

contract on indices a and e:

$$\nabla_e R_{bd} + \nabla^a R_{abde} - \nabla_d R_{be} = 0$$

If $R_{ab} = 0$, then the first and third terms vanish and we have $\nabla^a R_{abde} = 0$.

3. (a)

$$g_{ab} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \rho^2 \end{pmatrix}$$

$$g^{ab} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \frac{1}{\rho^2} \end{pmatrix}$$

Only non-zero $\partial_c g_{ab}$ is $\partial_0 g_{\varphi\varphi}$, so only possible non-zero Γ^a_{bc} are $\Gamma^0_{\varphi\varphi}$ and $\Gamma^\varphi_{0\varphi} = \Gamma^\varphi_{\varphi 0}$:

$$\begin{aligned}
\Gamma^0_{\varphi\varphi} &= \frac{1}{2} g^{00} (\cancel{\partial_\varphi g_{\varphi 0}} + \cancel{\partial_\varphi g_{0\varphi}} - \partial_0 g_{\varphi\varphi}) \\
&= \frac{1}{2} \partial_0 \rho^2 \\
&= \rho \rho'
\end{aligned}$$

$$
\begin{aligned}
\Gamma^{\varphi}_{0\varphi} &= \frac{1}{2}\, g^{\varphi\varphi}(\partial_0 g_{\varphi\varphi} + \cancel{\partial_\varphi g_{\varphi 0}} - \cancel{\partial_\varphi g_{0\varphi}}) \\
&= \frac{1}{2}\,\frac{1}{\rho^2}\,\partial_0 \rho^2 \\
&= \frac{\rho'}{\rho}
\end{aligned}
$$

(b) Independent components of the Riemann tensor are R_{0z0z}, $R_{0z0\varphi}$, $R_{0zz\varphi}$, $R_{0\varphi 0\varphi}$, $R_{0\varphi z\varphi}$, and $R_{z\varphi z\varphi}$.

But since none of the Christoffel symbols has a z component, those components of R_{abcd} which have a z component are zero. Only possible non-zero component is $R_{0\varphi 0\varphi}$:

$$
\begin{aligned}
R_{0\varphi 0}{}^{\varphi} &= \cancel{\partial_\varphi \Gamma^{\varphi}_{00}} - \partial_0 \Gamma^{\varphi}_{\varphi 0} + \cancel{\Gamma^{e}_{00}\Gamma^{\varphi}_{\varphi e}} - \Gamma^{e}_{\varphi 0}\Gamma^{\varphi}_{0e} \\
&= -\partial_0 \Gamma^{\varphi}_{\varphi 0} - \Gamma^{\varphi}_{\varphi 0}\Gamma^{\varphi}_{0\varphi} \\
&= -\partial_0 \left(\frac{\rho'}{\rho}\right) - \frac{\rho'^2}{\rho^2} \\
&= -\frac{\rho''}{\rho} + \frac{\rho'^2}{\rho^2} - \frac{\rho'^2}{\rho^2} \\
&= -\frac{\rho''}{\rho}
\end{aligned}
$$

(c) Flat $\Leftrightarrow$ $R_{abc}{}^{d} = 0$:

$$
\rho'' = 0 \quad \Rightarrow \quad \rho(t) = At + B
$$

The resulting metric is, up to a coordinate transformation, just 3-dimensional Minkowski space. Independent continuous symmetries are:

(i) 2 boosts

(ii) 1 rotation

(iii) 3 translations

Hence there are 6 independent continuous symmetries.

4. (a) Consider the Lagrangian:

$$
L = g_{ab}U^a U^b = -\Big(1 - \frac{2m}{r}\Big)\,\dot{t}^2 + \Big(1 - \frac{2m}{r}\Big)^{-1} \dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\varphi}^2
$$

In the equatorial plane, $\theta = \pi/2$ and $\dot{\theta} = 0$.

Euler–Lagrange eqn for t:

$$\left(\frac{\partial L}{\partial \dot{t}}\right)^{\cdot} - \frac{\partial L}{\partial t} = 0$$

$$\left(\left(1 - \frac{2m}{r}\right)\dot{t}\right)^{\cdot} = 0$$

$$\left(1 - \frac{2m}{r}\right)\dot{t} = E$$

Similarly E-L eqn for φ:

$$\left(2r^2\dot{\varphi}\right)^{\cdot} = 0$$

$$r^2\dot{\varphi} = J$$

Substitute $\dot{t} = \frac{E}{1-\frac{2m}{r}}$, $\dot{\varphi} = \frac{J}{r^2}$ into L and use the fact that $U^a U_a = 0$ for null geodesics:

$$\left(1 - \frac{2m}{r}\right)^{-1} E^2 = \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 + \frac{J^2}{r^2}$$

$$\dot{r} = \sqrt{E^2 - \frac{J^2(r-2m)}{r^3}}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \sqrt{\frac{r^4E^2}{J^2} - r(r-2m)}$$

Set $u = \frac{2m}{r}$

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = -\frac{2m}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\varphi}$$

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi} = \sqrt{\frac{4m^2E^2}{J^2} - u^2 + u^3}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} = \frac{1}{2}\frac{-2u+3u^2}{\sqrt{\frac{4m^2E^2}{J^2} - u^2 + u^3}}\frac{\mathrm{d}u}{\mathrm{d}\varphi} = -u + \frac{3}{2}u^2$$

Hence

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{3}{2}u^2$$

(b) For $u = \frac{2}{3}$,

LHS:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{2}{3}$$

RHS:

$$\frac{3}{2}u^2 = \frac{2}{3}$$

This describes a circular photon orbit with radius $r = 3m$.

(c) For $u = \frac{2}{3} - \frac{2}{1+\cosh\varphi}$,

LHS:

$$\begin{aligned}
\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u &= -\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\frac{2}{1+\cosh\varphi} + \frac{2}{3} - \frac{2}{1+\cosh\varphi} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varphi}\frac{2\sinh\varphi}{(1+\cosh\varphi)^2} + \frac{2}{3} - \frac{2}{1+\cosh\varphi} \\
&= \frac{2\cosh\varphi}{(1+\cosh\varphi)^2} - \frac{4\sinh^2\varphi}{(1+\cosh\varphi)^3} + \frac{2}{3} - \frac{2}{1+\cosh\varphi} \\
&= \frac{2\cosh\varphi}{(1+\cosh\varphi)^2} - \frac{4(\cosh\varphi - 1)}{(1+\cosh\varphi)^2} + \frac{2}{3} - \frac{2}{1+\cosh\varphi} \\
&= \frac{2 - 4\cosh\varphi}{(1+\cosh\varphi)^2} + \frac{2}{3}
\end{aligned}$$

RHS:

$$\begin{aligned}
\frac{3}{2}u^2 &= \frac{3}{2}\left(\frac{2}{3} - \frac{2}{1+\cosh\varphi}\right)^2 \\
&= \frac{3}{2}\left(\frac{4}{9} - \frac{8}{3}\frac{1}{1+\cosh\varphi} + \frac{4}{(1+\cosh\varphi)^2}\right) \\
&= \frac{2}{3} + \frac{2 - 4\cosh\varphi}{(1+\cosh\varphi)^2}
\end{aligned}$$

When $\varphi = \cosh^{-1} 2$, $u = 0$ which means $r = \infty$. On the other hand, when $\varphi = \infty$, $u = \frac{2}{3}$ which is the orbit considered in part (b). Hence the photon starts from infinity and asymptotes to the circular orbit of part (b), in a spiral-shaped orbit.