NATIONAL UNIVERSITY OF SINGAPORE

# PC4248 RELATIVITY

(Semester I: AY 2014–15)

Time Allowed: 2 Hours

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## INSTRUCTIONS TO CANDIDATES

1. Write your matriculation number only. **Do not write your name.**
2. This examination paper contains **FOUR** questions and comprises **THREE** printed pages.
3. Answer **ALL** questions.
4. Please start each question on a new page.
5. This is a **closed book** examination.
6. Students are allowed to bring in one A4-sized double-sided help sheet.

1. The 4-acceleration of a particle is defined to be

$$A^a = \frac{\mathrm{d}U^a}{\mathrm{d}\tau},$$

where $U^a$ is the particle's 4-velocity and $\tau$ its proper time.

(a) Show that

$$A^a = \gamma^4 \left( a^j u_j, (a^j u_j) u^i + \frac{a^i}{\gamma^2} \right),$$

where $u^i$ is the particle's 3-velocity, $a^i$ its 3-acceleration and $\gamma \equiv 1/\sqrt{1 - u^i u_i}$.

(b) Show that

$$A^a A_a = \gamma^6 \left( (a^i u_i)^2 + \frac{a^i a_i}{\gamma^2} \right).$$

(c) Show that $A^a U_a = 0$.

2. Consider a two-dimensional surface with metric

$$\mathrm{d}s^2 = \frac{4}{(1+r^2)^2} \left( \mathrm{d}r^2 + r^2 \, \mathrm{d}\phi^2 \right),$$

where $0 < r < \infty$ and $0 \leq \phi < 2\pi$.

(a) Show that the non-zero Christoffel symbols are

$$\Gamma^r_{rr} = -\frac{2r}{1+r^2}, \qquad \Gamma^r_{\phi\phi} = \frac{r(r^2-1)}{1+r^2}, \qquad \Gamma^\phi_{r\phi} = -\frac{r^2-1}{r(1+r^2)}.$$

(b) A vector $V^a = (V^r, V^\phi)$ is parallel transported along a curve $r = R$, from the point $\phi = 0$ to $\phi = \phi_0$. (Here, $R$ and $\phi_0$ are constants.) Find the value of $R$ for which $V^a$ remains unchanged as it is parallel transported along the curve.

(c) Is the curve you found in part (b) a geodesic? Explain your reasoning.

3. (a) Starting from the following form of the Riemann tensor:

$$R_{abcd} = g_{de}(\partial_b \Gamma^e_{ac} - \partial_a \Gamma^e_{bc} + \Gamma^f_{ac}\Gamma^e_{bf} - \Gamma^f_{bc}\Gamma^e_{af}),$$

show that, in geodesic coordinates,

$$R_{abcd} = \tfrac{1}{2}(\partial_a\partial_d\, g_{bc} + \partial_b\partial_c\, g_{ad} - \partial_b\partial_d\, g_{ac} - \partial_a\partial_c\, g_{bd}).$$

(b) Using the result of part (a), verify the identity

$$R_{abcd} + R_{acdb} + R_{adbc} = 0.$$

(c) Still in geodesic coordinates, verify the Bianchi identity

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0.$$

4. Suppose space-time has an extra fifth dimension. The analogue of the Schwarzschild solution in this case is

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2 + \cos^2\theta\,\mathrm{d}\psi^2),$$

where $\psi$ is the extra (angular) coordinate.

(a) Use the Lagrangian formalism to derive the geodesic equation for $\theta$, and show that it is possible for a geodesic to lie entirely in the equatorial plane $\theta = \pi/2$.

(b) Derive the remaining geodesic equations governing the motion of a timelike particle in the equatorial plane.

(c) By writing the geodesic equation for $r$ in the form

$$\dot{r}^2 = E^2 - V(r)$$

and analyzing the function $V(r)$ for $r > 0$, show that there are *no* stable timelike particle orbits in the equatorial plane.

(ET)

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