NATIONAL UNIVERSITY OF SINGAPORE

PC4248 RELATIVITY

(Semester I: AY 2015-16)

Time Allowed: 2.0 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. **Do not write your name.**
2. This examination paper contains **THREE (3)** questions and comprises **THREE (3)** printed pages.
3. Students are required to answer **ALL** questions.
4. Students should write the answers for each question on a new page.
5. This is a **CLOSED** book examination.
6. Students are allowed to bring in **ONE** A4-sized (both sides) sheet of notes.

Question 1: Physics in Flat Spacetime. [30=10+10+10]

Let S and S' be inertial frames of reference, with S' moving in the $+x$-direction with speed β relative to S. The coordinates (t, x, y, z) and (t', x', y', z'), in frames S and S' respectively, are related by

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\Theta & -\sinh\Theta & 0 & 0 \\ -\sinh\Theta & \cosh\Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \qquad \text{where} \qquad \beta \equiv \tanh\Theta\,.$$

(a) Given two arbitrary four-vectors $\mathbf{A}$ and $\mathbf{B}$ in frame S, the scalar product between $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{A}\cdot\mathbf{B} \equiv \eta_{\mu\nu}A^{\mu}B^{\nu}\,,$$

where $\eta_{\mu\nu}$ are Minkowski metric tensor components. Show that $\mathbf{A}\cdot\mathbf{B}$ is invariant in all inertial frames.

(b) A massive particle has a 3-velocity (u_x, u_y, u_z) as measured in S, and (u'_x, u'_y, u'_z) as measured in S'. Derive the relation between these 3-velocities in terms of Θ.

(c) Show that, if the speed of a massive particle is less than 1 in any one inertial frame, then it is less than 1 in every inertial frame.

Question 2: Tensors. [30=5+10+5+10]

The spacetime concerned in this problem is of dimension N and is described by the metric tensor $g_{\mu\nu}$ in the coordinate system $\{x^{\alpha}\}$.

The covariant derivative, a generalization of partial derivative in arbitrary spacetime, of a contravariant vector V^{α} is a rank-2 tensor of type $(1,1)$ given by

$$\nabla_{\beta}V^{\alpha} = \partial_{\beta}V^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}V^{\gamma}\,,$$

where the Christoffel symbols, in terms of metric tensor components, are given by

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\,g^{\alpha\sigma}\left(\partial_{\beta}g_{\gamma\sigma} + \partial_{\gamma}g_{\sigma\beta} - \partial_{\sigma}g_{\beta\gamma}\right)\,.$$

(a) Given two tensors $\mathbf{A}$ and $\mathbf{B}$ of *arbitrary* types, show that when the commutator $(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu})$ is applied on $\mathbf{AB}$ (all indices have been suppressed), we have

$$(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu})\,(\mathbf{AB}) = \mathbf{A}\left[(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu})\,\mathbf{B}\right] + \left[(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu})\,\mathbf{A}\right]\mathbf{B}\,.$$

(b) Show that when the commutator is applied to a contravariant vector V^{α}, we have

$$(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu})\,V^{\alpha} = R^{\alpha}{}_{\beta\mu\nu}V^{\beta}\,,$$

where $R^{\alpha}{}_{\beta\mu\nu}$ is the Riemann curvature tensor given by

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}_{\beta\nu} - \partial_{\nu}\Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\gamma}_{\beta\nu} - \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\sigma}_{\beta\mu}\,.$$

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(c) Given the fact that $V^\alpha V_\alpha$ is a scalar, show that when the commutator is applied on a covariant vector V_α, we have

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V_\beta = -R^\alpha_{\ \beta\mu\nu} V_\alpha .$$

(d) Using the results above, evaluate the following expression:

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \mathbf{W} = ??$$

where $\mathbf{W}$ is a rank-2 tensor of type $(0, 2)$. Hence, deduce the following expression:

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \mathbf{Z} = ??$$

where $\mathbf{Z}$ is a rank-5 tensor of type $(3, 2)$.

Question 3: Physics in Curved Spacetime. [40=15+10+5+5+5]

The Schwarzschild-de Sitter solution, in coordinates $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$, given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2\right)^{-1} \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\phi^2 ,$$

is a one-parameter generalization of the Schwarzschild solution. This solution describes the spacetime geometry of a spherical gravitating object with mass M in four dimensions with a positive cosmological constant Λ.

(a) Derive the equations of motion for a massless particle moving on the equatorial plane $(\theta = \pi/2)$ in the Schwarzschild-de Sitter spacetime. In particular, use the equations of motion to show that

$$\tilde{E} = \alpha(r) \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \tilde{V}(r) .$$

What are the expressions for $\tilde{E}$, $\alpha(r)$ and $\tilde{V}(r)$?

(b) Determine the radial coordinate of possible circular orbits (if any) in the Schwarzschild-de Sitter spacetime. For each possible circular orbit, determine the nature of the orbit (stable or unstable). Substantiate your claim with analytical calculations.

(c) Determine all non-vanishing independent Ricci tensor components of the Schwarzschild-de Sitter spacetime.

(d) Calculate the Ricci scalar of the Schwarzschild-de Sitter spacetime.

(e) Give a mathematical expression to determine the *possible* radial coordinate for the event horizon of the Schwarzschild-de Sitter spacetime. Explain.

(KHCM)

END OF PAPER