NATIONAL UNIVERSITY OF SINGAPORE

PC4248 RELATIVITY

(Semester I: AY 2016-17)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. **Do not write your name.**
2. This examination paper contains **THREE (3)** questions and comprises **FOUR (4)** printed pages.
3. Students are required to answer **ALL** questions.
4. Students should write the answers for each question on a new page.
5. This is a **CLOSED** book examination. You are allowed however to bring in **ONE** A4-sized (both sides) sheet of notes.
6. Specific permitted devices: non-programmable calculators.

Question 1

[30=15+15]

The spacetime concerned in this problem is of dimension N and is described by the metric tensor $g_{\mu\nu}$ in the coordinate system $\{x^\alpha\}$.

The covariant differentiation, a generalization of partial differentiation in arbitrary spacetime, of a contravariant vector V^α is a rank-2 tensor of type $(1,1)$ given by

$$\nabla_\beta V^\alpha = \partial_\beta V^\alpha + \Gamma^\alpha_{\beta\gamma} V^\gamma ,$$

where the Christoffel symbols, in terms of metric tensor components, are given by

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma} \left(\partial_\beta g_{\gamma\sigma} + \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\beta\gamma} \right) .$$

(a) Demanding that covariant differentiation of a contravariant vector is a rank-2 tensor of type $(1,1)$, show that the Christoffel symbols are transformed as the following rule:

$$\Gamma'^\alpha_{\beta\gamma} = \frac{\partial x'^\alpha}{\partial x^\sigma} \frac{\partial x^\mu}{\partial x'^\beta} \frac{\partial x^\nu}{\partial x'^\gamma} \Gamma^\sigma_{\mu\nu} + \frac{\partial x'^\alpha}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x'^\beta \, \partial x'^\gamma} .$$

(b) The geodesic equation is given as follows:

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} = 0 .$$

Suppose $K_{\mu\nu}$ is a symmetric tensor satisfying

$$\nabla_\lambda K_{\mu\nu} + \nabla_\mu K_{\nu\lambda} + \nabla_\nu K_{\lambda\mu} = g_{\lambda\mu} V_\nu + g_{\mu\nu} V_\lambda + g_{\nu\lambda} V_\mu ,$$

for some covariant vector V_μ, which in turn satisfies

$$\nabla_\lambda V_\mu + \nabla_\mu V_\lambda = k g_{\lambda\mu} ,$$

for some constant k. Show that the quantity

$$Q = k L^2 \tau^2 - 2 L u^\mu V_\mu \tau + K_{\mu\nu} u^\mu u^\nu ,$$

where u^μ is the four-velocity and $L = \dfrac{1}{2} g_{\mu\nu} u^\mu u^\nu$, is a constant for any massive particle following a geodesic.

Question 2 **[40=10+10+10+10]**

The flat Robertson-Walker metric, in coordinates $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) .$$

It describes the spacetime geometry of a homogeneous, isotropic and spatially flat universe where $a(t)$ is the scale factor whose time dependence describes the expansion of the universe. The source of curvature is a homogeneous and isotropic perfect fluid of matter, radiation and vacuum energy whose energy-momentum tensor is given by

$$T^{\mu\nu} = \left[\rho(t) + P(t)\right] u^\mu u^\nu + P(t) g^{\mu\nu} - \frac{\Lambda}{8\pi}\, g^{\mu\nu} ,$$

where $u^\mu = (1, 0, 0, 0)$ is the fluid four-velocity, Λ is the cosmological constant, $\rho(t)$ and $P(t)$ are the energy density and pressure.

(a) The Christoffel symbols, in terms of metric tensor components, are given by

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\, g^{\alpha\sigma}\left(\partial_\beta g_{\gamma\sigma} + \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\beta\gamma}\right) .$$

Show that the only non-vanishing components of the Christoffel symbols are

$$\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = a(t)\,\frac{\mathrm{d}a(t)}{\mathrm{d}t} \qquad \text{and} \qquad \Gamma^1_{10} = \Gamma^2_{20} = \Gamma^3_{30} = \frac{1}{a(t)}\,\frac{\mathrm{d}a(t)}{\mathrm{d}t} .$$

(b) The Ricci tensor $R_{\alpha\beta}$ is defined as follows:

$$R_{\beta\nu} \equiv R^\alpha{}_{\beta\alpha\nu} = g^{\alpha\mu} R_{\alpha\beta\mu\nu} , \qquad R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\beta\mu} .$$

Show that all off-diagonal components of the Ricci tensor are zero and calculate the diagonal components of the Ricci tensor.

(c) The Einstein field equation in four-dimensional spacetime is given by

$$R^{\mu\nu} = 8\pi\left(T^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu} T\right) , \qquad T \equiv g_{\mu\nu} T^{\mu\nu} .$$

Find the 00- and 11-components of the Einstein field equation. Guess the 22- and 33-components of the Einstein field equation. Explain your answers.

(d) Local conservation theorem of energy-momentum dictates that

$$\nabla_\mu T^{\mu\nu} = 0 .$$

Using local conservation of energy-momentum, show that

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} + 3\,\frac{\rho(t)}{a(t)}\,\frac{\mathrm{d}a(t)}{\mathrm{d}t} + 3\,\frac{P(t)}{a(t)}\,\frac{\mathrm{d}a(t)}{\mathrm{d}t} = 0 .$$

Question 3 **[30=5+5+5+5+10]**

The Schwarzschild solution, in coordinates $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$, is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \, d\theta^2 + r^2 \sin^2\theta \, d\phi^2 .$$

It describes the spacetime geometry of a spherical gravitating object with mass M in four dimensions. Kruskal coordinates (U, V) are defined as follows:

$$\left.\begin{aligned} U &= \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh\left(\frac{t}{4M}\right) \\ V &= \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh\left(\frac{t}{4M}\right) \end{aligned}\right\} \quad r > 2M$$

$$\left.\begin{aligned} U &= \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \sinh\left(\frac{t}{4M}\right) \\ V &= \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \cosh\left(\frac{t}{4M}\right) \end{aligned}\right\} \quad r < 2M$$

The Schwarzschild solution, in coordinates $(x^0, x^1, x^2, x^3) = (U, V, \theta, \phi)$, in either the region with $r < 2M$ or that with $r > 2M$ is given by

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} \left(dV^2 - dU^2\right) + r^2 \, d\theta^2 + r^2 \sin^2\theta \, d\phi^2 .$$

(a) A plot of V versus U, i.e. V is the vertical axis and U is the horizontal axis, is called a Kruskal diagram. Show that the lines of constant r and t are hyperbolas and straight lines through the origin in the UV plane, respectively.

(b) Give a Kruskal diagram with clear labels of the following:

- **(i)** constant-r curves with values $r = 0, M, 2M, 3M, \infty$
- **(ii)** constant-t curves with values $t = -\infty, -3M, -2M, -M, 0, M, 2M, 3M, \infty$

(c) Show that the coordinate V is always timelike and the coordinate U is always spacelike.

(d) Show that worldlines of radial photons are always 45° with the vertical in the Kruskal diagram.

(e) Professor Einstein falls feet first into a Schwarzschild black hole looking down his feet. Analyze the following questions with a Kruskal diagram.

- **(i)** Can he see his feet when his head is crossing the horizon? If so, at what radius does he see them all?
- **(ii)** Does he see his feet hit the singularity assuming that he remains intact until his head reaches the singularity?
- **(iii)** Is there ever a moment when he cannot see his feet?

KHCM

END OF PAPER