NATIONAL UNIVERSITY OF SINGAPORE

PC4248 RELATIVITY

(Semester I: AY 2017-18)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. **Do not write your name.**
2. This examination paper contains **THREE (3)** questions and comprises **THREE (3)** printed pages.
3. Students are required to answer **ALL** questions.
4. Students should write the answers for each question on a new page.
5. This is a **CLOSED** book examination. However, you are allowed to bring in **ONE** A4-sized help sheet.
6. Specific permitted devices: non-programmable calculators.

Question 1 [25=20+5]

The metric of a two-dimensional space, in coordinates $(x^1, x^2) = (r, \phi)$, is given by

$$ds^2 = dr^2 + \lambda^2 r^2 \, d\phi^2 \,, \qquad 0 \le r < \infty \,, \qquad 0 \le \phi < 2\pi \,,$$

where λ is a real constant. A contravariant vector with components $(V^1, V^2) = (V_0^r, V_0^\phi)$ is located at the point $(x^1, x^2) = (r_0, 0)$. Suppose it is parallel transported around a circle $r = r_0$ to the point $(x^1, x^2) = (r_0, 2\pi/\lambda)$.

(a) Calculate the components of the parallel-transported vector.

(b) Are circles of constant r geodesics? Explain.

Question 2 [30=20+10]

Consider the Newtonian limit of general relativity, given by the metric:

$$ds^2 = -\,(1 + 2\Phi)\, dt^2 + (1 - 2\Phi)\, \delta_{ij}\, dx^i\, dx^j \,,$$

where $\Phi = \Phi(x^i)$ is the Newtonian gravitational potential, and $i, j = 1, 2, 3$. The only non-vanishing Christoffel symbols, to first-order in Φ, are

$$\Gamma^0_{0i} = \partial_i \Phi \,, \qquad \Gamma^i_{00} = \partial^i \Phi \,, \qquad \Gamma^i_{jk} = \partial^i \Phi \, \delta_{jk} - \partial_j \Phi \, \delta^i_k - \partial_k \Phi \, \delta^i_j \,.$$

(a) For a slowly moving perfect fluid with pressure $P_0 \ll \rho_0$, show that Einstein's field equation with a non-zero cosmological constant Λ:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \,,$$

reduces to the modified Poisson equation:

$$\nabla^2 \Phi = 4\pi\rho - \Lambda \,,$$

where $\nabla^2 = \delta_{ij}\partial_i\partial_j$ and ρ_0 is the proper density of the fluid.

(b) Hence, show that the corresponding Newtonian gravitational potential of a spherically symmetric mass M centered at the origin can be written as:

$$\Phi = -\frac{M}{r} - \frac{\Lambda r^2}{6} \,,$$

where $r^2 = \delta_{ij} x^i x^j$. Give a physical interpretation of this result.

Question 3 **[45=10+15+10+10]**

In the 2014 science fiction movie *Interstellar*, the planet Miller is circularly orbiting around a rotating black hole Gargantua. The spacetime geometry around a rotating black hole is described by the Kerr metric and is given, in the coordinates $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$, by:

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 - \frac{4Mar\sin^2\theta}{\Sigma}\mathrm{d}t\,\mathrm{d}\phi + \Sigma\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right) + \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta\,\mathrm{d}\phi^2\,,$$

where

$$\Sigma = r^2 + a^2\cos^2\theta\,, \qquad \Delta = r^2 - 2Mr + a^2\,.$$

Here, M and a are parameters related to the mass and rotation of the black hole.

(a) Derive the geodesic equation for θ and show that it is possible for a timelike geodesic to lie entirely in the equatorial plane $\theta = \pi/2$. Indicate clearly the initial conditions for these equatorial timelike geodesics.

(b) Derive the remaining geodesic equations governing the timelike geodesics in the equatorial plane. Hence, or otherwise, show that the geodesic equation for r can be written in the form

$$\frac{1}{2}\left(e^2 - 1\right) = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{M}{r} + \frac{\ell^2 + a^2(1 - e^2)}{2r^2} - \frac{M(\ell - ae)^2}{r^3}\,,$$

where e and ℓ are two conserved quantities along the timelike geodesics.

(c) Defining the angular speed of a massive particle as $\Omega \equiv \mathrm{d}\phi/\mathrm{d}t$, show that the period of equatorial circular orbits for a massive particle is

$$T = \left|\frac{2\pi}{\Omega}\right| = \sqrt{\frac{4\pi^2}{M}}\,r^{3/2} \pm 2\pi a\,.$$

Note that in the limit $a \to 0$, this becomes Kepler's third law. Give a reason why there are two branches of answer.

(d) In the *Interstellar*, the gravitational field is so strong that a severe time dilation is experienced on the surface of the Miller planet: one hour there is seven years on Earth. It is believed that Gargantua is to be rotating maximally and Miller is very close to the event horizon of Gargantua. Show that the innermost stable circular orbit, $r = r_c$, satisfies the following equations:

$$\begin{cases} Mr_c^2 - \alpha^2 r_c + 3M\beta^2 = 0 \\ 2Mr_c^2 - 3\alpha^2 r_c + 12M\beta^2 = 0 \end{cases}.$$

What are the expressions for α^2 and β^2? Hence, determine the values for e and ℓ for the innermost stable circular orbit at $r_c = M$ if the black hole rotates maximally, where $a = M$.

KHCM

END OF PAPER