The answer for certain questions in this document is incomplete. Would you like to help us complete it? If yes, Please send your suggested answers to nus.physoc@gmail.com. Thanks! ☺

Question 1

$$\Delta\phi = 2\int_{r_1}^{\infty} \frac{dr}{r^2\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)}}$$

$$w = \frac{b}{r} \quad \Rightarrow \quad dr = -\frac{r^2}{b}dw$$

$$\Delta\phi = 2\int_{0}^{w_1} \frac{dw}{b\sqrt{\frac{1}{b^2} - \frac{w^2}{b^2}\left(1 - \frac{2M}{b}w\right)}}$$

$$= 2\int_{0}^{w_1} \frac{dw}{\sqrt{1 - w^2\left(1 - \frac{2M}{b}w\right)}}$$

$$= 2\int_{0}^{w_1} \frac{dw}{\sqrt{1 - \frac{2M}{b}w}\sqrt{\frac{1}{1 - \frac{2M}{b}w} - w^2}}$$

$$\approx 2\int_{0}^{w_1} \frac{\left(1 + \frac{M}{b}w\right)dw}{\sqrt{1 + \frac{2M}{b}w - w^2}}$$

$$= 2\int_{0}^{w_1} \frac{\left(1 + \frac{M}{b}w\right)dw}{\sqrt{1 + \frac{M^2}{b^2} - \left(w - \frac{M}{b}\right)^2}}$$

$$\approx 2\int_{0}^{w_1} \frac{1 + \frac{M}{b}w}{\sqrt{1 - \left(w - \frac{M}{b}\right)^2}}dw$$

$$= 2\int_{0}^{1+\frac{M}{b}} \frac{1}{\sqrt{1 - \left(w - \frac{M}{b}\right)^2}}dw + \frac{2M}{b}\int_{0}^{1+\frac{M}{b}} \frac{w}{\sqrt{1 - \left(w - \frac{M}{b}\right)^2}}dw$$

$$= 2\int_{0}^{1+\frac{M}{b}} \frac{1 + \frac{4M^2}{b^2}}{\sqrt{1 - \left(w - \frac{M}{b}\right)^2}}dw + \frac{2M}{b}\int_{0}^{1+\frac{M}{b}} \frac{w - \frac{M}{b}}{\sqrt{1 - \left(w - \frac{M}{b}\right)^2}}dw$$

$$\approx 2\int_{0}^{1+\frac{M}{b}} \frac{1}{\sqrt{1 - \left(w - \frac{M}{b}\right)^2}}dw - \frac{2M}{b}\left[\sqrt{1 - \left(w - \frac{M}{b}\right)^2}\right]_0^{1+\frac{M}{b}}$$

$$= 2\left[\sin^{-1}\left(w - \frac{M}{b}\right)\right]_0^{1+\frac{M}{b}} + \frac{2M}{b}\sqrt{1 - \frac{M^2}{b^2}}$$

$$\approx \pi - 2\sin^{-1}\left(-\frac{M}{b}\right) + \frac{2M}{b}$$

$$\approx \pi + \frac{2M}{b} + \frac{2M}{b}$$

$$= \pi + \frac{4M}{b}$$

$$\therefore \delta\phi_{def} = \Delta\phi - \pi = \frac{4M}{b} = \frac{4GM}{bc^2}$$

In the case of grazing the edge of the sun,

$$\delta\phi_{def} = \frac{4GM_\odot}{c^2 R_\odot}$$

Question 2

$$\frac{dr}{d\tau} = \pm\sqrt{e^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right)}$$

Proper time after the particle crossed the horizon,

$$\tau = -\int_{2M}^{0} \frac{dr}{\sqrt{e^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right)}}$$

For maximum proper time, $e^2 = 0$, $l^2 = 0$:

$$\tau = \int_0^{2M} \frac{dr}{\sqrt{\frac{2M}{r} - 1}}$$

$$r = u^2, \qquad dr = 2u\,du$$

$$\tau = \int_0^{\sqrt{2M}} \frac{2u\,du}{\sqrt{\frac{2M}{u^2} - 1}}$$

$$= \int_0^{\sqrt{2M}} \frac{2u^2}{\sqrt{2M - u^2}}du$$

$$= -\left[2u\sqrt{2M - u^2}\right]_0^{\sqrt{2M}} + \int_0^{\sqrt{2M}} 2\sqrt{2M - u^2}\,du$$

$$= \int_0^{\sqrt{2M}} 2\sqrt{2M - u^2}\,du$$

$u = \sqrt{2M}\sin\theta\,, \qquad du = \sqrt{2M}\cos\theta\,d\theta$

$$\tau = 2\int_0^{\frac{\pi}{2}} \sqrt{2M - 2M\sin^2\theta}\,\sqrt{2M}\cos\theta\,d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} 2M\cos^2\theta\,d\theta$$
$$= 4M\int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{\cos 2\theta}{2}\,d\theta$$
$$= 4M\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}$$
$$= 4M\left(\frac{\pi}{M}\right)$$
$$= \pi M$$

$\therefore \tau_{max} = \pi M$

Question 3 (a)

$ds^2 = -2du\,dv + a^2(u)dx^2 + b^2(u)dy^2$

$$L = \frac{d\tau}{d\sigma} = \sqrt{-\frac{ds^2}{d\sigma^2}} = \sqrt{2\frac{du}{d\sigma}\frac{dv}{d\sigma} - a^2\left(\frac{dx}{d\sigma}\right)^2 - b^2\left(\frac{dy}{d\sigma}\right)^2}$$

$$\frac{\partial L}{\partial u} = \frac{d}{d\sigma}\frac{\partial L}{\partial\left(\frac{du}{d\sigma}\right)}$$
$$\left[-2aa'\left(\frac{dx}{d\sigma}\right)^2 - 2bb'\left(\frac{dy}{d\sigma}\right)^2\right]\frac{1}{2}\frac{d\sigma}{d\tau} = \frac{d}{d\tau}\left(2\frac{dv}{d\sigma}\frac{1}{2}\frac{d\sigma}{d\tau}\right)$$
$$\frac{d^2v}{d\tau^2} = -aa'\left(\frac{dx}{d\tau}\right)^2 - bb'\left(\frac{dy}{d\tau}\right)^2, \qquad (1)$$

$$\frac{\partial L}{\partial v} = \frac{d}{d\sigma}\frac{\partial L}{\partial\left(\frac{dv}{d\sigma}\right)}$$
$$0 = \frac{d}{d\tau}\left(2\frac{du}{d\sigma}\frac{1}{2}\frac{d\sigma}{d\tau}\right)$$
$$\frac{d^2u}{d\tau^2} = 0, \qquad (2)$$

$$\frac{\partial L}{\partial x} = \frac{d}{d\sigma}\frac{\partial L}{\partial\left(\frac{dx}{d\sigma}\right)}$$

$$0 = \frac{d}{d\sigma}\left[\left(-2a^2\frac{dx}{d\sigma}\right)\frac{1}{2}\frac{d\sigma}{d\tau}\right]$$

$$= \frac{d}{d\tau}\left(-a^2\frac{dx}{d\tau}\right)$$

$$= -2aa'\frac{du}{d\tau}\frac{dx}{d\tau} - a^2\frac{d^2x}{d\tau^2}$$

$$\frac{d^2x}{d\tau^2} = -\frac{2a'}{a}\frac{du}{d\tau}\frac{dx}{d\tau}, \qquad (3)$$

By symmetry,

$$\frac{d^2y}{d\tau^2} = -\frac{2b'}{b}\frac{du}{d\tau}\frac{dy}{d\tau}, \qquad (4)$$

So the non-zero Christoffel symbols are

$$\Gamma^v_{xx} = aa'$$

$$\Gamma^v_{yy} = bb'$$

$$\Gamma^x_{ux} = \Gamma^x_{xu} = \frac{a'}{a}$$

$$\Gamma^y_{uy} = \Gamma^y_{yu} = \frac{b'}{b}$$

Question 3 (b)

$$R_{\alpha\beta} = \partial_\epsilon\Gamma^\epsilon_{\alpha\beta} - \partial_\beta\Gamma^\epsilon_{\alpha\epsilon} + \Gamma^\epsilon_{\alpha\beta}\Gamma^\rho_{\epsilon\rho} - \Gamma^\epsilon_{\alpha\rho}\Gamma^\rho_{\beta\epsilon}$$

$$R_{uu} = -\partial_u\Gamma^x_{ux} - \partial_u\Gamma^y_{uy} - \Gamma^x_{ux}\Gamma^x_{ux} - \Gamma^y_{uy}\Gamma^y_{uy}$$

$$= \left(\frac{a'^2}{a^2} - \frac{a''}{a}\right) + \left(\frac{b'^2}{b^2} - \frac{b''}{b}\right) - \left(\frac{a'}{a}\right)^2 - \left(\frac{b'}{b}\right)^2$$

$$= -\left(\frac{a''}{a} + \frac{b''}{b}\right)$$

$$R_{vv} = 0, \qquad R_{xx} = 0, \qquad R_{yy} = 0$$

$$R_{uv} = R_{vu} = 0, \qquad R_{xy} = R_{yx} = 0$$

$$R_{ux} = R_{xu} = 0, \qquad R_{uy} = R_{yu} = 0$$

$$R_{vx} = R_{xv} = 0, \qquad R_{vy} = R_{yv} = 0$$

$\therefore$ The only non-zero Ricci curvature tensor,

$$R_{uu} = -\left(\frac{a''}{a} + \frac{b''}{b}\right)$$

Question 3 (c)

$R_{\alpha\beta} = 0$

$$R_{uu} = -\left(\frac{a''}{a} + \frac{b''}{b}\right) = -\frac{1}{a}\frac{d^2a}{du^2} - \frac{1}{b}\frac{d^2b}{du^2} = 0$$

$$-\frac{1}{a}\frac{d^2a}{du^2} - \frac{1}{b}\frac{d^2b}{du^2} = 0$$

$$\frac{1}{a}\frac{d^2a}{du^2} + \frac{1}{b}\frac{d^2b}{du^2} = 0$$

$$\frac{ab}{a}\frac{d^2a}{du^2} + \frac{ab}{b}\frac{d^2b}{du^2} = 0$$

$$b\frac{d^2a}{du^2} + a\frac{d^2b}{du^2} = 0$$

$$\frac{d^2}{du^2}(ab) = 0$$

$$\frac{d}{du}(ab) = k_1$$

$\therefore a(u)b(u) = k_1 u + k_2$

where k_1, k_2 are constants.

Question 3 (d)

Solutions provided by:

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