**NATIONAL UNIVERSITY OF SINGAPORE**

PC4248 RELATIVITY

(Semester I: AY 2010-11)

Time Allowed: 2 Hours

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## INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **3** questions and comprises **3** printed pages.
2. Answer ALL 3 questions.
3. Answers to the questions are to be written in the answer books.
4. All questions carry equal marks.
5. This is a CLOSED BOOK examination.

# Useful formulas

(a) You may use the natural units: $G=1$ and $c=1$.

(b) The metric for Minkowski spacetime with Cartesian coordinates is

$\eta_{\mu\nu} = \mathrm{diag}\{-1,1,1,1\}$.

(c) The Christoffel symbols are defined as

$$\Gamma^{\alpha}_{\beta\gamma} \equiv \frac{1}{2} g^{\alpha\delta}\left(\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma}\right)$$

(d) The Riemann curvature tensor is defined as

$$R^{\alpha}{}_{\beta\gamma\delta} \equiv \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\gamma\varepsilon}\Gamma^{\varepsilon}_{\beta\delta} - \Gamma^{\alpha}_{\delta\varepsilon}\Gamma^{\varepsilon}_{\beta\gamma}$$

(e) Schwarzschild geometry is given by

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

(f) Kerr geometry is given by

$$ds^2 = -\left(1-\frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2$$

with $a \equiv \frac{J}{M}$, $\quad \rho^2 \equiv r^2 + a^2\cos^2\theta$, $\quad \Delta \equiv r^2 - 2Mr + a^2$

(g) Table of integrals

$$\int \frac{u du}{\sqrt{1-u^2}} = -\sqrt{1-u^2}$$

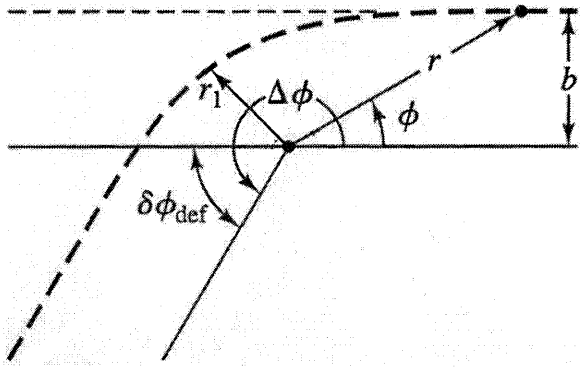
$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u$$

$$\int \frac{du}{u\sqrt{1-u^2}} = -\ln\frac{1+\sqrt{1-u^2}}{u}$$

$$\int \frac{du}{(1+u)\sqrt{1-u^2}} = -\sqrt{\frac{1-u}{1+u}}$$

$$\int \frac{u^2 du}{\sqrt{1-u^2}} = -\frac{1}{2}u\sqrt{1-u^2} + \frac{1}{2}\sin^{-1} u$$

1. Calculate the deflection of light, $\delta\phi_{\text{def}} \equiv \Delta\phi - \pi$, when a light ray is grazing the edge of the Sun.




Express your results in terms of solar mass $M_\odot$ and solar radius $R_\odot$.

2. Once across the event horizon of a Schwarzschild black hole with mass $M$, what is the *longest* proper time the observer can spend before being destroyed in the singularity?

3. An *exact* gravitational plane wave solution to Einstein's field equation has the line metric

$$ds^2 = -2dudv + a^2(u)dx^2 + b^2(u)dy^2,$$

where $a$ and $b$ are functions of $u$.

(a) Calculate the Christoffel symbols.

(b) Calculate the Ricci curvature tensors.

(c) Use Einstein's equation in vacuum to derive equations obeyed by $a(u)$ and $b(u)$.

(d) Show that an exact solution can be found, in which both $a$ and $b$ are determined in terms of an *arbitrary* function $f(u)$.

[*hint:* $2dudv = dudv + dvdu$.]

(WQh)

----- End of Paper ------