Question 1 (a)

A boost in $\frac{1}{\sqrt{2}}(0,1,1,0)$ direction requires us to rotate the plane by $45°$, and thus transform the Lorentz transformation matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -\frac{1}{\sqrt{2}}v\gamma & \frac{1}{\sqrt{2}}\gamma & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}}v\gamma & \frac{1}{\sqrt{2}}\gamma & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\frac{1}{\sqrt{2}}v\gamma & -\frac{1}{\sqrt{2}}v\gamma & 0 \\ -\frac{1}{\sqrt{2}}v\gamma & \frac{1}{2}\gamma + \frac{1}{2} & \frac{1}{2}\gamma - \frac{1}{2} & 0 \\ -\frac{1}{\sqrt{2}}v\gamma & \frac{1}{2}\gamma - \frac{1}{2} & \frac{1}{2}\gamma + \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So the transformation of the coordinates,

$$\begin{pmatrix} \gamma & -\frac{1}{\sqrt{2}}v\gamma & -\frac{1}{\sqrt{2}}v\gamma & 0 \\ -\frac{1}{\sqrt{2}}v\gamma & \frac{1}{2}(\gamma + 1) & \frac{1}{2}(\gamma - 1) & 0 \\ -\frac{1}{\sqrt{2}}v\gamma & \frac{1}{2}(\gamma - 1) & \frac{1}{2}(\gamma + 1) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma t - \frac{1}{\sqrt{2}}v\gamma x - \frac{1}{\sqrt{2}}v\gamma y \\ -\frac{1}{\sqrt{2}}v\gamma t + \frac{1}{2}(\gamma + 1)x + \frac{1}{2}(\gamma - 1)y \\ -\frac{1}{\sqrt{2}}v\gamma t + \frac{1}{2}(\gamma - 1)x + \frac{1}{2}(\gamma + 1)y \\ z \end{pmatrix}$$

$$t' = \gamma\left(t - v\frac{x + y}{\sqrt{2}}\right)$$

$$x' = \gamma\left(-\frac{vt}{\sqrt{2}} + \frac{x + y}{2}\right) + \frac{x - y}{2}$$

$$y' = \gamma\left(-\frac{vt}{\sqrt{2}} + \frac{x + y}{2}\right) - \frac{x - y}{2}$$

$$z' = z$$

Question 1 (b)

When at a fixed time $t = 0$,

$$x' = \gamma\left(\frac{x+y}{2}\right) + \frac{x-y}{2}$$

We set $x_1 = 0, x_2 = L, y = 0$. Then

$$\Delta x' = x_2' - x_1' = \gamma\frac{L}{2} + \frac{L}{2} = \frac{L}{2}(\gamma + 1)$$

$$\therefore L = \frac{L_*}{2}(\gamma + 1)$$

Question 2 (a)

We know that

$$T'^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}T^{\alpha}{}_{\beta}$$

So then we have

$$T'^{\mu_1,\mu_2,\dots,\mu_r}{}_{\nu_1,\nu_2,\dots,\nu_s} = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha}}\frac{\partial x'^{\mu_2}}{\partial x^{\alpha}}\cdots\frac{\partial x'^{\mu_r}}{\partial x^{\alpha}}\right)\left(\frac{\partial x^{\beta}}{\partial x'^{\nu_1}}\frac{\partial x^{\beta}}{\partial x'^{\nu_2}}\cdots\frac{\partial x^{\beta}}{\partial x'^{\nu_s}}\right)T^{\alpha_1,\alpha_2,\dots,\alpha_r}{}_{\beta_1,\beta_2,\dots,\beta_s}$$

Question 2 (b)

We assume

$$\begin{aligned}
\nabla'_m V'^n &= \frac{\partial x^a}{\partial x'^m}\frac{\partial x'^n}{\partial x^b}\nabla_a V^b \\
&= \frac{\partial x^a}{\partial x'^m}\frac{\partial x'^n}{\partial x^b}(\partial_a V^b + \Gamma^b_{ac}V^c) \\
&= \frac{\partial x^a}{\partial x'^m}\frac{\partial x'^n}{\partial x^b}\frac{\partial V^b}{\partial x^a} + \frac{\partial x^a}{\partial x'^m}\frac{\partial x'^n}{\partial x^b}\Gamma^b_{ac}V^c, \qquad (1)
\end{aligned}$$

We also know that

$$\begin{aligned}
\nabla'_m V'^n &= \partial'_m V'^n + \Gamma'^n_{mp}V'^p \\
&= \frac{\partial x^a}{\partial x'^m}\frac{\partial}{\partial x^a}\left(\frac{\partial x'^n}{\partial x^b}V^b\right) + \Gamma'^n_{mp}\frac{\partial x'^p}{\partial x^c}V^c \\
&= \frac{\partial x^a}{\partial x'^m}\frac{\partial^2 x'^n}{\partial x^a \partial x^b}V^b + \frac{\partial x^a}{\partial x'^m}\frac{\partial x'^n}{\partial x^b}\frac{\partial V^b}{\partial x^a} + \Gamma'^n_{mp}\frac{\partial x'^p}{\partial x^c}V^c, \qquad (2)
\end{aligned}$$

(1) = (2),

$$\frac{\partial x^a}{\partial x'^m}\frac{\partial x'^n}{\partial x^b}\frac{\partial V^b}{\partial x^a} + \frac{\partial x^a}{\partial x'^m}\frac{\partial x'^n}{\partial x^b}\Gamma^b_{ac}V^c = \frac{\partial x^a}{\partial x'^m}\frac{\partial^2 x'^n}{\partial x^a \partial x^b}V^b + \frac{\partial x^a}{\partial x'^m}\frac{\partial x'^n}{\partial x^b}\frac{\partial V^b}{\partial x^a} + \Gamma'^n_{mp}\frac{\partial x'^p}{\partial x^c}V^c$$

$$\frac{\partial x^a}{\partial x'^m}\frac{\partial x'^n}{\partial x^b}\Gamma^b_{ac}V^c = \frac{\partial x^a}{\partial x'^m}\frac{\partial^2 x'^n}{\partial x^a \partial x^b}V^b + \Gamma'^n_{mp}\frac{\partial x'^p}{\partial x^c}V^c$$

$$\Gamma'^n_{mp}\frac{\partial x'^p}{\partial x^c}V^c = \frac{\partial x^a}{\partial x'^m}\frac{\partial x'^n}{\partial x^b}\Gamma^b_{ac}V^c - \frac{\partial x^a}{\partial x'^m}\frac{\partial^2 x'^n}{\partial x^a \partial x^b}V^b$$

$$\Gamma'^{n}_{mp}V^c = \frac{\partial x^c}{\partial x'^p}\frac{\partial x^a}{\partial x'^m}\frac{\partial x'^n}{\partial x^b}\Gamma^b_{ac}V^c - \frac{\partial x^c}{\partial x'^p}\frac{\partial x^a}{\partial x'^m}\frac{\partial^2 x'^n}{\partial x^a \partial x^b}\frac{\partial x^b}{\partial x^c}V^c$$
$$= \frac{\partial x^c}{\partial x'^p}\frac{\partial x^a}{\partial x'^m}\frac{\partial x'^n}{\partial x^b}\Gamma^b_{ac}V^c - \frac{\partial x^b}{\partial x'^p}\frac{\partial x^a}{\partial x'^m}\frac{\partial^2 x'^n}{\partial x^a \partial x^b}V^c$$

$$\therefore \Gamma'^{n}_{mp} = \frac{\partial x^c}{\partial x'^p}\frac{\partial x^a}{\partial x'^m}\frac{\partial x'^n}{\partial x^b}\Gamma^b_{ac} - \frac{\partial x^b}{\partial x'^p}\frac{\partial x^a}{\partial x'^m}\frac{\partial^2 x'^n}{\partial x^a \partial x^b}$$

Question 2 (c) (i)

$$V'^b\partial'_b W'^a - W'^b\partial'_b V'^a = \frac{\partial x'^b}{\partial x^\nu}V^\nu\frac{\partial x^\nu}{\partial x'^b}\frac{\partial}{\partial x^\nu}\left(\frac{\partial x'^a}{\partial x^\mu}W^\mu\right) - \frac{\partial x'^b}{\partial x^\nu}W^\nu\frac{\partial x^\nu}{\partial x'^b}\frac{\partial}{\partial x^\nu}\left(\frac{\partial x'^a}{\partial x^\mu}V^\mu\right)$$
$$= V^\nu\left(\frac{\partial^2 x'^a}{\partial x^\nu \partial x^\mu}W^\mu + \frac{\partial x'^a}{\partial x^\mu}\frac{\partial W^\mu}{\partial x^\nu}\right) - W^\nu\left(\frac{\partial^2 x'^a}{\partial x^\nu \partial x^\mu}V^\mu + \frac{\partial x'^a}{\partial x^\mu}\frac{\partial V^\mu}{\partial x^\nu}\right)$$
$$= V^\nu\frac{\partial^2 x'^a}{\partial x^\nu \partial x^\mu}W^\mu - W^\nu\frac{\partial^2 x'^a}{\partial x^\nu \partial x^\mu}V^\mu + V^\nu\frac{\partial x'^a}{\partial x^\mu}\frac{\partial W^\mu}{\partial x^\nu} - W^\nu\frac{\partial x'^a}{\partial x^\mu}\frac{\partial V^\mu}{\partial x^\nu}$$
$$= \frac{\partial x'^a}{\partial x^\mu}\left(V^\nu\frac{\partial W^\mu}{\partial x^\nu} - W^\nu\frac{\partial V^\mu}{\partial x^\nu}\right)$$
$$= \frac{\partial x'^a}{\partial x^\mu}\left(V^\nu\partial_\nu W^\mu - W^\nu\partial_\nu V^\mu\right)$$

∴ It is a tensor.

Question 2 (c) (ii)

$$\Gamma'^{c}_{ab}B'^{ab} = \left(\frac{\partial x^\nu}{\partial x'^b}\frac{\partial x^\mu}{\partial x'^a}\frac{\partial x'^c}{\partial x^\lambda}\Gamma^\lambda_{\mu\nu} - \frac{\partial x^\nu}{\partial x'^b}\frac{\partial x^\mu}{\partial x'^a}\frac{\partial^2 x'^c}{\partial x^\mu \partial x^\nu}\right)\left(\frac{\partial x'^a}{\partial x^\mu}\frac{\partial x'^b}{\partial x^\nu}B^{\mu\nu}\right)$$
$$= \frac{\partial x'^c}{\partial x^\lambda}\Gamma^\nu_{\mu\lambda}B^{\mu\nu} - \frac{\partial^2 x'^c}{\partial x^\mu \partial x^\nu}B^{\mu\nu}$$

Since $B^{\nu\mu} = -B^{\mu\nu}$,

$$\frac{\partial^2 x'^c}{\partial x^\mu \partial x^\nu}B^{\mu\nu} = \frac{\partial^2 x'^c}{\partial x^\mu \partial x^\nu}B^{\nu\mu} = -\frac{\partial^2 x'^c}{\partial x^\mu \partial x^\nu}B^{\mu\nu} = 0$$

$$\therefore \Gamma'^{c}_{ab}B'^{ab} = \frac{\partial x'^c}{\partial x^\lambda}\Gamma^\nu_{\mu\lambda}B^{\mu\nu}, \text{ it is a tensor.}$$

Question 3 (a)

$ds^2 = y^p\, dx^2 + x^q\, dy^2$

$$L = \frac{d\tau}{d\sigma} = \sqrt{\left(-\frac{ds}{d\sigma}\right)^2} = \sqrt{-y^p\left(\frac{dx}{d\sigma}\right)^2 - x^q\left(\frac{dy}{d\sigma}\right)^2}$$

$$\frac{\partial L}{\partial x} = \frac{d}{d\sigma}\frac{\partial L}{\partial\left(\frac{dx}{d\sigma}\right)}$$

$$-qx^{q-1}\left(\frac{dy}{d\sigma}\right)^2\frac{1}{2}\frac{d\sigma}{d\tau} = \frac{d}{d\tau}\left(-2y^p\frac{dx}{d\sigma}\frac{1}{2}\frac{d\sigma}{d\tau}\right)$$

$$\frac{1}{2}qx^{q-1}\left(\frac{dy}{d\tau}\right)^2 = \frac{d}{d\tau}\left(y^p\frac{dx}{d\tau}\right)$$

$$\frac{1}{2}qx^{q-1}\left(\frac{dy}{d\tau}\right)^2 = py^{p-1}\frac{dx}{d\tau}\frac{dy}{d\tau} + y^p\frac{d^2x}{d\tau^2}$$

$$\frac{d^2x}{d\tau^2} = \frac{1}{2}q\frac{x^{q-1}}{y^p}\left(\frac{dy}{d\tau}\right)^2 - \frac{p}{y}\frac{dx}{d\tau}\frac{dy}{d\tau}, \qquad (1)$$

By symmetry,

$$\frac{d^2y}{d\tau^2} = \frac{1}{2}p\frac{y^{p-1}}{x^q}\left(\frac{dx}{d\tau}\right)^2 - \frac{q}{x}\frac{dx}{d\tau}\frac{dy}{d\tau}, \qquad (2)$$

∴ The non-vanishing Christoffel symbols,

$$\Gamma^x_{yy} = -\frac{1}{2}q\frac{x^{q-1}}{y^p}$$

$$\Gamma^y_{xx} = -\frac{1}{2}p\frac{y^{p-1}}{x^q}$$

$$\Gamma^x_{xy} = \Gamma^x_{yx} = \frac{1}{2}\frac{p}{y}$$

$$\Gamma^y_{xy} = \Gamma^y_{yx} = \frac{1}{2}\frac{q}{x}$$

Question 3 (b)

$R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\gamma\epsilon}\Gamma^{\epsilon}_{\beta\delta} - \Gamma^{\alpha}_{\delta\epsilon}\Gamma^{\epsilon}_{\beta\gamma}$

$R^{x}{}_{xxx} = 0, \qquad R^{y}{}_{yyy} = 0$

$R^{x}{}_{yyy} = 0, \qquad R^{y}{}_{xxx} = 0$

$R^{x}{}_{xxy} = \Gamma^{x}_{xy}\Gamma^{y}_{xy} - \Gamma^{x}_{yy}\Gamma^{y}_{xx} = \frac{1}{4}\frac{pq}{xy} - \frac{1}{4}\frac{pq}{xy} = 0$

$R^{x}{}_{xyx} = \Gamma^{x}_{yy}\Gamma^{y}_{xx} - \Gamma^{x}_{xy}\Gamma^{y}_{xy} = \frac{1}{4}\frac{pq}{xy} - \frac{1}{4}\frac{pq}{xy} = 0$

$R^{x}{}_{yxx} = -R^{x}{}_{xxy} - R^{x}{}_{xyx} = 0$

$R^{x}{}_{xyy} = 0$

$$\begin{aligned} R^{x}{}_{yxy} &= \partial_x \Gamma^{x}_{yy} - \partial_y \Gamma^{x}_{yx} - \Gamma^{x}_{yx}\Gamma^{x}_{yx} - \Gamma^{x}_{yy}\Gamma^{y}_{yx} \\ &= -\frac{1}{2}q(q-1)\frac{x^{q-2}}{y^p} + \frac{1}{2}\frac{p}{y^2} - \frac{1}{4}\frac{p^2}{y^2} + \frac{1}{4}q^2\frac{x^{q-2}}{y^p} \\ &= \frac{q}{2}\frac{x^{q-2}}{y^p}\left(1-\frac{q}{2}\right) + \frac{p}{2}\frac{1}{y^2}\left(1-\frac{p}{2}\right) \end{aligned}$$

$R^{x}{}_{yyx} = -R^{x}{}_{xyy} - R^{x}{}_{yxy} = \frac{q}{2}\frac{x^{q-2}}{y^p}\left(\frac{q}{2}-1\right) + \frac{p}{2}\frac{1}{y^2}\left(\frac{p}{2}-1\right)$

$$\begin{aligned} R^{y}{}_{xxy} &= \partial_x \Gamma^{y}_{xy} - \partial_y \Gamma^{y}_{xx} + \Gamma^{y}_{xx}\Gamma^{x}_{xy} + \Gamma^{y}_{xy}\Gamma^{y}_{xy} \\ &= -\frac{1}{2}\frac{q}{x^2} + \frac{1}{2}p(p-1)\frac{y^{p-2}}{x^q} - \frac{1}{4}p^2\frac{y^{p-2}}{x^q} + \frac{1}{4}\frac{q^2}{x^2} \\ &= \frac{p}{2}\frac{y^{p-2}}{x^q}\left(\frac{p}{2}-1\right) + \frac{1}{2}\frac{q}{x^2}\left(\frac{q}{2}-1\right) \end{aligned}$$

$R^{y}{}_{yxx} = 0$

$R^{y}{}_{xyx} = -R^{y}{}_{xxy} - R^{y}{}_{yxx} = \frac{p}{2}\frac{y^{p-2}}{x^q}\left(1-\frac{p}{2}\right) + \frac{1}{2}\frac{q}{x^2}\left(1-\frac{q}{2}\right)$

$R^{y}{}_{xyy} = 0$

$R^{y}{}_{yxy} = \Gamma^{y}_{xx}\Gamma^{x}_{yy} - \Gamma^{y}_{yx}\Gamma^{x}_{yx} = \frac{1}{4}\frac{pq}{xy} - \frac{1}{4}\frac{pq}{xy} = 0$

$R^{y}{}_{yyx} = -R^{y}{}_{xyy} - R^{y}{}_{yxy} = 0$

∴ the non-zero components of the Riemann Curvature tensor,

$R^{x}{}_{yxy} = \frac{q}{2}\frac{x^{q-2}}{y^p}\left(1-\frac{q}{2}\right) + \frac{1}{2}\frac{p}{y^2}\left(1-\frac{p}{2}\right)$

$R^{y}{}_{xyx} = \frac{p}{2}\frac{y^{p-2}}{x^q}\left(1-\frac{p}{2}\right) + \frac{1}{2}\frac{q}{x^2}\left(1-\frac{q}{2}\right)$

$R^{x}{}_{yyx} = \frac{q}{2}\frac{x^{q-2}}{y^p}\left(\frac{q}{2}-1\right) + \frac{1}{2}\frac{p}{y^2}\left(\frac{p}{2}-1\right)$

$R^{y}{}_{xxy} = \frac{p}{2}\frac{y^{p-2}}{x^q}\left(\frac{p}{2}-1\right) + \frac{1}{2}\frac{q}{x^2}\left(\frac{q}{2}-1\right)$

Question 3 (c)

When $p = q = 0$, since

$$\begin{pmatrix} y^p & 0 \\ 0 & x^q \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Question 4 (a)

For photons, $\underset{\sim}{u} \cdot \underset{\sim}{u} = 0$

$$0 = -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2 \sin^2\theta \left(\frac{d\phi}{d\tau}\right)^2$$

Using the implications of Killing vectors,

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}, \qquad l = r^2 \sin^2\theta \frac{d\phi}{d\tau}$$

Setting $\theta = \frac{\pi}{2}$, we get

$$0 = -\left(1 - \frac{2M}{r}\right)^{-1} e^2 + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + \frac{l^2}{r^2}$$

$$\left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 = \left(1 - \frac{2M}{r}\right)^{-1} e^2 - \frac{l^2}{r^2}$$

$$\dot{r}^2 = e^2 - \left(1 - \frac{2M}{r}\right)\frac{l^2}{r^2} = e^2\left[1 - \left(1 - \frac{2M}{r}\right)\frac{l^2}{e^2r^2}\right] = E^2\left[1 - \frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right)\right]$$

Question 4 (b)

We consider the case when the photon is neither moving towards outwards from the black hole,

$$0 = 1 - \frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right)$$

$$b^2 = \frac{r^2}{1 - \frac{2M}{r}} = \frac{r^3}{r - 2M}$$

$$0 = \frac{d}{dr}\left(\frac{r^3}{r - 2M}\right) = \frac{3r^2}{r - 2M} - \frac{r^3}{(r - 2M)^2} = \frac{2r^2(r - 3M)}{(r - 2M)^2}, \qquad r = 3M$$

This means that $b^2 = \frac{27M^3}{M} = 27M^2$. If $b^2 < 27M^2$ it will plunge into the black hole; $b^2 > 27M^2$ it will be deflected, but remain in circular motion if $b^2 = 27M^2$.

Solutions provided by:

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