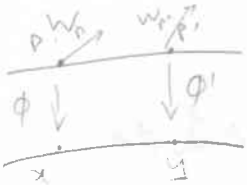
PC4274 2008-2009 Final

1) Let W be a smooth vector field for which we wish to find the Lie derivative at $p \in M$. Let $\gamma_t(p)$ be the local one parameter group of local diffeomorphisms (LOPGOLD) induced by V



Then let $\gamma_t(p) = p'$, and then

$$\mathcal{L}_V W|_p = \lim_{t \to 0} \frac{1}{t}[\gamma_{-t*} W_{p'} - W_p]$$

Similarly for a one form, $\mathcal{L}_V \omega|_p = \lim_{t \to 0} \frac{1}{t}[\gamma_t^* \omega_{p'} - \omega_p]$

For a general tensor, R,

$\mathcal{L}_V R|_p = \lim_{t \to 0} \frac{1}{t}[\gamma_{\pm t}^* R_{p'} - R_p]$, where $\gamma_{\pm t}^*$ mean we push forward components of $R_{p'}$ that are vectors by γ_{-t*} and pull back components of R_p that are forms via γ_t^*.

The exterior derivative $d: A^k(M) \to A^{k+1}(M)$ and satisfies the following properties.

(1) d is linear

(2) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ for $\alpha \in A^k(M), \beta \in A^l(M)$

(3) $d^2 = 0$

(4) If $f \in C^\infty(M)$, then df is the ordinary differential.

1a) $\mathcal{L}_V d\omega = d[d\omega(V)] + \underbrace{(d(d\omega))}_{d^2=0}(V)$

$d\mathcal{L}_V \omega = \underbrace{d^2[\omega(V)]}_{=0} + d(d\omega(V)) = \mathcal{L}_V d\omega$

1b) When $k=0$, ω is a function, so

$$\begin{aligned}
LHS &= \mathcal{L}_V d\omega \\
&= \mathcal{L}_V \left(\frac{\partial \omega}{\partial x^i} dx^i\right) \\
&= \mathcal{L}_V\left(\frac{\partial \omega}{\partial x^i}\right) dx^i + \frac{\partial \omega}{\partial x^i} \mathcal{L}_V(dx^i) \\
&= V\left(\frac{\partial \omega}{\partial x^i}\right) dx^i + \frac{\partial \omega}{\partial x^i} d\left[\langle dx^i, V^j \frac{\partial}{\partial x^j}\rangle\right] \\
&= V^j \frac{\partial^2 \omega}{\partial x^j \partial x^i} dx^i + \frac{\partial \omega}{\partial x^i} dV^i \\
&= V^i \frac{\partial^2 \omega}{\partial x^i \partial x^j} dx^j + \frac{\partial \omega}{\partial x^i} \frac{\partial V^i}{\partial x^j} dx^j
\end{aligned}$$

$$\begin{aligned}
RHS &= d\mathcal{L}_V(\omega) \\
&= d\, V(\omega) \\
&= d\left(V^i \frac{\partial \omega}{\partial x^i}\right) \\
&= \frac{\partial V^i}{\partial x^j} \frac{\partial \omega}{\partial x^i} dx^j + V^i \frac{\partial^2 \omega}{\partial x^j \partial x^i} dx^j \\
&= V^i \frac{\partial^2 \omega}{\partial x^i \partial x^j} dx^j + \frac{\partial V^i}{\partial x^j} \frac{\partial \omega}{\partial x^i} dx^j \\
&= LHS.
\end{aligned}$$

1b) When $k=1$, w is a one-form, so $w = w_i dx^i$

$$\begin{aligned}
LHS &= \mathcal{L}_V dw \\
&= \mathcal{L}_V dw_i \wedge dx^i \\
&= \mathcal{L}_V \frac{\partial w_i}{\partial x^j} dx^j \wedge dx^i \\
&= \mathcal{L}_V\left(\frac{\partial w_i}{\partial x^j}\right) dx^j \wedge dx^i + \frac{\partial w_i}{\partial x^j}(\mathcal{L}_V dx^j) \wedge dx^i + \frac{\partial w_i}{\partial x^j} dx^j \wedge \mathcal{L}_V dx^i \\
&= V\left(\frac{\partial w_i}{\partial x^j}\right) dx^j \wedge dx^i + \frac{\partial w_i}{\partial x^j} d[dx^j(V)] \wedge dx^i + \frac{\partial w_i}{\partial x^j} dx^j \wedge d[dx^i(V)] \\
&= V^k \frac{\partial^2 w_i}{\partial x^k \partial x^j} dx^j \wedge dx^i + \frac{\partial w_i}{\partial x^j}\frac{\partial V^j}{\partial x^k} dx^k \wedge dx^i + \frac{\partial w_i}{\partial x^j}\frac{\partial V^i}{\partial x^k} dx^j \wedge dx^k \\
&= \left[V^k \frac{\partial^2 w_i}{\partial x^k \partial x^j} + \frac{\partial w_i}{\partial x^k}\frac{\partial V^k}{\partial x^j} + \frac{\partial w_k}{\partial x^j}\frac{\partial V^k}{\partial x^i}\right] dx^j \wedge dx^i
\end{aligned}$$

$$\begin{aligned}
RHS &= d\mathcal{L}_V w_i dx^i \\
&= d[\mathcal{L}_V(w_i) dx^i + w_i \mathcal{L}_V(dx^i)] \\
&= d[V(w_i) dx^i + w_i dV^i] \\
&= d\left[V^j \frac{\partial w_i}{\partial x^j} dx^i + w_i dV^i\right] \\
&= \frac{\partial V^j}{\partial x^k}\frac{\partial w_i}{\partial x^j} dx^k \wedge dx^i + V^j \frac{\partial^2 w_i}{\partial x^k \partial x^j} dx^k \wedge dx^i + dw_i \wedge dV^i \\
&= \frac{\partial V^k}{\partial x^j}\frac{\partial w_i}{\partial x^k} dx^j \wedge dx^i + V^k \frac{\partial^2 w_i}{\partial x^j \partial x^k} dx^j \wedge dx^i + \frac{\partial w_i}{\partial x^j}\frac{\partial V^i}{\partial x^k} dx^j \wedge dx^k \\
&= \left[\frac{\partial V^k}{\partial x^j}\frac{\partial w_i}{\partial x^k} + V^k \frac{\partial^2 w_i}{\partial x^j \partial x^k} + \frac{\partial w_k}{\partial x^j}\frac{\partial V^k}{\partial x^i}\right] dx^j \wedge dx^i \\
&= LHS.
\end{aligned}$$

2a) $V = V^1 \frac{\partial}{\partial x^1} + V^2 \frac{\partial}{\partial x^2}$

$\mathcal{L}_V(dx^1) = d[\langle dx^1, V\rangle] = dV^1$, $\mathcal{L}_V(dx^2) = dV^2$

$$\begin{aligned}
\mathcal{L}_V g &= (\mathcal{L}_V dx^1) \otimes dx^1 + dx^1 \otimes \mathcal{L}_V dx^1 + (\mathcal{L}_V dx^2) \otimes dx^2 \\
&\quad + dx^2 \otimes \mathcal{L}_V(dx^2) \\
&= \frac{\partial V^1}{\partial x^1} dx^1 \otimes dx^1 + \frac{\partial V^1}{\partial x^2} dx^2 \otimes dx^1 + \frac{\partial V^1}{\partial x^1} dx^1 \otimes dx^1 \\
&\quad + \frac{\partial V^1}{\partial x^2} dx^1 \otimes dx^2 + \frac{\partial V^2}{\partial x^1} dx^1 \otimes dx^2 + \frac{\partial V^2}{\partial x^2} dx^2 \otimes dx^2 \\
&\quad + \frac{\partial V^2}{\partial x^1} dx^2 \otimes dx^1 + \frac{\partial V^2}{\partial x^2} dx^2 \otimes dx^2 \\
&= 2\frac{\partial V^1}{\partial x^1} dx^1 \otimes dx^1 + \left(\frac{\partial V^1}{\partial x^2} + \frac{\partial V^2}{\partial x^1}\right) dx^1 \otimes dx^2 + \left(\frac{\partial V^1}{\partial x^2} + \frac{\partial V^2}{\partial x^1}\right) dx^2 \otimes dx^1 \\
&\quad + 2\frac{\partial V^2}{\partial x^2} dx^2 \otimes dx^2
\end{aligned}$$

$\mathcal{L}_V g = 0 \Rightarrow$ ① $\frac{\partial V^1}{\partial x^1} = 0 = \frac{\partial V^2}{\partial x^2}$, $\frac{\partial V^1}{\partial x^2} + \frac{\partial V^2}{\partial x^1} = 0$ ②

① implies: $V^1 = V^1(x^2)$, $V^2 = V^2(x^1)$

② implies: $\frac{\partial V^1}{\partial x^2}(x^2) = -\frac{\partial V^2}{\partial x^1}(x^1)$

The left hand side is a function of x^2, while the right hand side is a function od x^1, so both sides are constant, say c.

So, $V^1 = cx^2 + a$

$V^2 = -cx^1 + b$, where $a, b \in \mathbb{R}$.

$V = (cx^2 + a)\frac{\partial}{\partial x^1} + (-cx^1 + b)\frac{\partial}{\partial x^2}$

2b) We can obtain three linearly independant killing vectors by setting one of the constants to 1 and the others to zero, and any arbitrary killing vector will be a linear combination of these three.

$a = 1$, $V_1 = \frac{\partial}{\partial x^1}$

$b = 1$, $V_2 = \frac{\partial}{\partial x^2}$

$c = 1$, $V_3 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}$

2c) $[V_1, V_2]f = \frac{\partial}{\partial x^1}\frac{\partial f}{\partial x^2} - \frac{\partial}{\partial x^2}\frac{\partial f}{\partial x^1}$

$= -\left(\frac{\partial}{\partial x^2}\frac{\partial f}{\partial x^1} - \frac{\partial}{\partial x^1}\frac{\partial f}{\partial x^2}\right) = 0$

$= -[V_2, V_1]f$.

$[V_1, V_3]f = \frac{\partial}{\partial x^1}\left(x^2\frac{\partial f}{\partial x^1} - x^1\frac{\partial f}{\partial x^2}\right) - \left(x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}\right)\frac{\partial f}{\partial x^1}$

$= x^2\frac{\partial^2 f}{\partial x^{12}} - \frac{\partial f}{\partial x^2} - x^1\frac{\partial^2 f}{\partial x^1 \partial x^2} - x^2\frac{\partial^2 f}{\partial x^{12}} + x^1\frac{\partial^2 f}{\partial x^2 \partial x^1}$

$= -\frac{\partial f}{\partial x^2}$

$[V_3, V_1]f = \left(x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}\right)\frac{\partial f}{\partial x^1} - \frac{\partial}{\partial x^1}\left(x^2\frac{\partial f}{\partial x^1} - x^1\frac{\partial f}{\partial x^2}\right)$

$= -\left(-\frac{\partial f}{\partial x^2}\right)$

$= \frac{\partial f}{\partial x^2} = [V_1, V_3]f$.

$[V_2, V_3] = V_2 \circ V_3 - V_3 \circ V_2 = -(V_3 \circ V_2 - V_2 \circ V_3) = -[V_3, V_2]$

$$[V_2, V_3]f = \frac{\partial}{\partial x^2}\left(x^2\frac{\partial f}{\partial x^1} - x^1\frac{\partial f}{\partial x^2}\right) - \left(x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}\right)\frac{\partial f}{\partial x^2}$$

$$= \frac{\partial f}{\partial x^1} + x^2\frac{\partial^2 f}{\partial x^2 \partial x^1} - x^1\frac{\partial^2 f}{\partial (x^2)^2} - x^2\frac{\partial^2 f}{\partial x^1 \partial x^2} + x^1\frac{\partial^2 f}{\partial (x^2)^2}$$

$$= \frac{\partial f}{\partial x^1} \Rightarrow [V_2, V_3] = \frac{\partial}{\partial x^1}$$

Now, we check the Jacobi Identity.

$$[V_1,[V_2,V_3]]f + [V_2,[V_3,V_1]]f + [V_3,[V_1,V_2]]f$$

$$= \underbrace{\left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}\right]}_{=0}f + \underbrace{\left[\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^2}\right]}_{=0}f + \left[x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}, 0\right]f$$

$= 0$ so V_1, V_2 and V_3 satisfy the Jacobi identity, so we have a lie algebra.

2d) The LOPGOLD induced by V_1 and V_2 sends points the the x^1 and x^2 direction respectively, while V_3 rotates points about the origin, so this should be the algebra of the $SE(2)$ group.

3a) Let $h = (u_1, v_1, \theta_1)$, $g = (u_2, v_2, \theta_2)$, $g \cdot h = (u_3, v_3, \theta_3)$. We will set $h = e$ at the end. For $f \in C^\infty(G)$,

$$L_{g*}\frac{\partial}{\partial u_1}f(h) = \frac{\partial}{\partial u_1} f \circ L_g(h)$$

$$= \frac{\partial}{\partial u_1} f(g \cdot h)$$

$$= \frac{\partial f}{\partial u_1}(u_3, v_3, \theta_3)$$

$$= \frac{\partial f}{\partial u_3}\frac{\partial u_3}{\partial u_1} + \frac{\partial f}{\partial v_3}\frac{\partial v_3}{\partial u_1} + \frac{\partial f}{\partial \theta_3}\frac{\partial \theta_3}{\partial u_1}$$

$$L_{g*}\frac{\partial}{\partial u_1} = \cos\theta_2 \frac{\partial}{\partial u_3} + \sin\theta_2 \frac{\partial}{\partial v_3}$$

$$L_{g*}\frac{\partial}{\partial v_1} = \frac{\partial u_3}{\partial v_1}\frac{\partial}{\partial u_3} + \frac{\partial v_3}{\partial v_1}\frac{\partial}{\partial v_3} + \frac{\partial \theta_3}{\partial v_1}\frac{\partial}{\partial \theta_3}$$

$$= -\sin\theta_2 \frac{\partial}{\partial u_3} + \cos\theta_2 \frac{\partial}{\partial v_3}$$

$$L_{g*}\frac{\partial}{\partial \theta_1} = \frac{\partial u_3}{\partial \theta_1}\frac{\partial}{\partial u_3} + \frac{\partial v_3}{\partial \theta_1}\frac{\partial}{\partial v_3} + \frac{\partial \theta_3}{\partial \theta_1}\frac{\partial}{\partial \theta_3} = \frac{\partial}{\partial \theta_3}$$

When $h = e$, $g \cdot h = g$, so

$$L_{g*}\frac{\partial}{\partial u_1} = \cos\theta_2 \frac{\partial}{\partial u_2} + \sin\theta_2 \frac{\partial}{\partial v_2}$$

$$L_{g*}\frac{\partial}{\partial v_1} = -\sin\theta_2 \frac{\partial}{\partial u_2} + \cos\theta_2 \frac{\partial}{\partial v_2}$$

$$L_{g*}\frac{\partial}{\partial \theta_1} = \frac{\partial}{\partial \theta_2}$$

3b) Let $w = a\,du_2 + b\,dv_2 + c\,d\theta_2$.

$$w_h = \langle du_1, \tfrac{\partial}{\partial u_1} \rangle_e$$
$$= \langle L_g^* w, \tfrac{\partial}{\partial u_1} \rangle_e$$
$$= \langle w, L_{g*} \tfrac{\partial}{\partial u_1} \rangle_g$$
$$= \langle a u_2 + b V_2 + c\theta_2, \cos\theta_2 \tfrac{\partial}{\partial u_2} + \sin\theta_2 \tfrac{\partial}{\partial v_2} \rangle_g$$
$$1 = a\cos\theta_2 + b\sin\theta_2 \quad (1)$$

$$0 = \langle du_1, \tfrac{\partial}{\partial v_1} \rangle.$$
$$= \langle w, L_{g*} \tfrac{\partial}{\partial v_1} \rangle$$
$$= \langle w, -\sin\theta_2 \tfrac{\partial}{\partial u_2} + \cos\theta_2 \tfrac{\partial}{\partial v_2} \rangle$$
$$= -a\sin\theta_2 + b\cos\theta_2$$
$$a\sin\theta_2 = b\cos\theta_2 \quad (2)$$

$$0 = \langle w, L_{g*} \tfrac{\partial}{\partial \theta_1} \rangle = c$$

From (1) & (2), $\cos\theta_2 = a\cos^2\theta_2 + a\sin^2\theta_2 = a$

so $b = \sin\theta_2$

and $w = \cos\theta_2\,du_2 + \sin\theta_2\,dV_2$

3c) We need to show that $L_g^* w_{g.h} = w_h$

Let $h = (u_1, v_1, \theta_1)$, $g = (u_2, V_2, \theta_2)$, $g.h = (u_3, V_3, \theta_3)$

$w_{g.h} = \cos\theta_3\,du_3 + \sin\theta_3\,dV_3$

$w_h = \cos\theta_1\,du_1 + \sin\theta_1\,dv_1$

Need to show that $\langle L_g^* w_{g.h}, X_h \rangle = \langle w_h, X_h \rangle$

where X_{h} is an arbitrary vector at h. Actually, we just need to show this for the basis vectors at h, $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \frac{\partial}{\partial \theta_1}$

$$\langle L_g^* w_{g.h}, \tfrac{\partial}{\partial u_1} \rangle = \langle w_{g.h}, L_{g*} \tfrac{\partial}{\partial u_1} \rangle$$
$$= \langle \cos\theta_3\,du_3 + \sin\theta_3\,dV_3, \cos\theta_2 \tfrac{\partial}{\partial u_3} + \sin\theta_2 \tfrac{\partial}{\partial V_3} \rangle$$
$$= \cos\theta_3\cos\theta_2 + \sin\theta_3\sin\theta_2$$
$$= \cos(\theta_3 - \theta_2)$$
$$= \cos(\theta_1)$$
$$= \langle \cos\theta_1\,du_1 + \sin\theta_1\,dv_1, \tfrac{\partial}{\partial u_1} \rangle$$
$$= \langle w_h, \tfrac{\partial}{\partial u_1} \rangle$$

$$\langle L_g^* w_{g\cdot n}, \tfrac{\partial}{\partial v_1}\rangle = \langle w_{g\cdot n}, L_{g*} \tfrac{\partial}{\partial v_1}\rangle$$

$$= \langle \cos\theta_3 du_3 + \sin\theta_3 dv_3, -\sin\theta_2 \tfrac{\partial}{\partial u_3} + \cos\theta_2 \tfrac{\partial}{\partial v_3}\rangle$$

$$= -\sin\theta_2 \cos\theta_3 + \sin\theta_3 \cos\theta_2$$

$$= \sin(\theta_3 - \theta_2)$$

$$= \sin\theta_1$$

$$= \langle \cos\theta_1 du_1 + \sin\theta_1 dv_1, \tfrac{\partial}{\partial v_1}\rangle$$

$$= \langle w_n, \tfrac{\partial}{\partial v_1}\rangle$$

$$\langle L_g^* w_{g\cdot n}, \tfrac{\partial}{\partial \theta_1}\rangle = \langle w_{g\cdot n}, \tfrac{\partial}{\partial \theta_3}\rangle = 0 = \langle \cos\theta_1 du_1 + \sin\theta_1 dv_1, \tfrac{\partial}{\partial \theta_1}\rangle$$

$$= \langle w_n, \tfrac{\partial}{\partial \theta_1}\rangle$$

So, $L_g^* w_{g\cdot n} = w_n$.