NATIONAL UNIVERSITY OF SINGAPORE

PC4274 MATHEMATICAL METHODS IN PHYSICS III

(Semester II: AY 2008-09)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **3** questions and comprises **3** printed pages including this page.
2. Answer **ALL THREE (3)** questions.
3. All questions carry equal marks.
4. Answers to the questions are to be written in the answer books.
5. This is a CLOSED BOOK examination.
6. One Help Sheet (A4 size, both sides) is allowed for this examination.

PC4274 - Mathematical Methods III

(1) On a manifold $\mathcal{M}$, one can introduce operators $\mathcal{L}_V$ (Lie-derivative with respect to the vector field V) and d (exterior derivative) on the space of forms. State briefly how they are defined. In connection with these operators, show the following.

(a) Given that the relation

$$\mathcal{L}_V\omega = d[\omega(V)] + d\omega(V)$$

holds for any k form, $\omega \in \mathcal{A}^k(\mathcal{M})$ and vector field $V \in \mathcal{X}(\mathcal{M})$, show that

$$\mathcal{L}_V d\omega = d\mathcal{L}_V\omega.$$

(b) Verify that the preceeding relation holds when $k = 0$ and $k = 1$.

(2) A Riemannian manifold $\mathcal{M} = \mathbf{R}^2$ with coordinates (x^1, x^2) is endowed with the metric tensor,

$$g = dx^1 \otimes dx^1 + dx^2 \otimes dx^2.$$

(a) Evaluate the Killing vectors V associated with this metric; *i.e.* obtain V by solving $\mathcal{L}_V g = 0$.

(b) Furnish all the linearly independent Killing vectors.

(c) Show that the Killing vectors obtained in part (2b) constitute a Lie algebra.

(d) To which Lie group is this algebra associated?

(**3**) On a group manifold $G = SE(2)$, the closure relation $\Psi : G \times G \rightarrow G$ is given by

$$\Psi(u_2, v_2, \theta_2; u_1, v_1, \theta_1) \equiv (u_2, v_2, \theta_2) \cdot (u_1, v_1, \theta_1) = (u_3, v_3, \theta_3)$$

where

$$\begin{aligned} u_3 &= u_2 + u_1 \cos\theta_2 - v_1 \sin\theta_2 \\ v_3 &= v_2 + u_1 \sin\theta_2 + v_1 \sin\theta_2 \\ \theta_3 &= \theta_1 + \theta_2. \end{aligned}$$

(**a**) By considering basis vectors at the tangent space of the identity, $T_e(G)$, furnish the set of left-invariant vector fields; *i.e.* compute $L_{g*}\frac{\partial}{\partial u_1}$, $L_{g*}\frac{\partial}{\partial v_1}$ and $L_{g*}\frac{\partial}{\partial \theta_1}$.

(**b**) On G, we can define left-invariant 1-forms ω associated with the basis 1-forms $du \in T_e{}^*(G)$ through $L_g{}^*\omega = du$. Show that the left-invariant 1-form at the point $g = (u_2, v_2, \theta_2)$ associated with the basis 1-form du_1 (at $g = e$) is given by

$$\omega = \cos\theta_2 du_2 + \sin\theta_2 dv_2.$$

(**c**) Show that the left-invariant one-form obtained in part (3b) is indeed left-invariant; *i.e.* show that $L_g{}^*\omega = \omega$ for all $g \in G$.

(KS)

**** **END OF PAPER** ****