P(4)74 2009-2010

1a) Let $V_H = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial p}$

$$w(V_H) = dH$$

$$(dx \otimes dp - dp \otimes dx)\left(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial p}\right) = \frac{\partial H}{\partial x} dx + \frac{\partial H}{\partial p} dp$$

$$\alpha\, dp - \beta\, dx = \alpha^2 x\, dx + p\, dp$$

So, $\alpha = p$, $\beta = -\alpha^2 x \Rightarrow V_H = p\frac{\partial}{\partial x} - \alpha^2 x \frac{\partial}{\partial p}$

Then, $\frac{dx}{dt} = p$, $\frac{dp}{dt} = -\alpha^2 x$. are the equations of the integral curves.

1b) $\frac{d^2x}{dt^2} = \frac{dp}{dt} = -\alpha^2 x$

$\frac{d^2p}{dt^2} = -\alpha^2 \frac{dx}{dt} = -\alpha^2 p.$

$\Rightarrow x(t) = A\cos\alpha t + B\sin\alpha t$

$p(t) = C\cos\alpha t + D\sin\alpha t$

Since $x(0) = x_0$, $p(0) = p_0$, $x(t) = x_0\cos\alpha t + B\sin\alpha t$

$p(t) = p_0\cos\alpha t + D\sin\alpha t$

Also, $p_0 = \frac{dx}{dt}\Big|_{t=0} = B\alpha \Rightarrow B = \frac{p_0}{\alpha}$

$-\alpha^2 x_0 = \frac{dp}{dt}\Big|_{t=0} = D\alpha \Rightarrow D = -\alpha x_0$.

So, $x(t) = x_0\cos\alpha t + \frac{p_0}{\alpha}\sin\alpha t$

$p(t) = p_0\cos\alpha t - \alpha x_0 \sin\alpha t$.

1c) Yes, because the integral curves are defined $\forall t \in \mathbb{R}$

1d) The integral curve is

$\gamma_{(x_0,p_0)}(t) = (x_0\cos\alpha t + \frac{p_0}{\alpha}\sin\alpha t,\ p_0\cos\alpha t - \alpha x_0 \sin\alpha t)$

The LOPGOLD is $\gamma_t(x_0,p_0) = \gamma_{(x_0,p_0)}(t)$

so $\gamma_t(x,p) = (x\cos\alpha t + \frac{p}{\alpha}\sin\alpha t,\ p\cos\alpha t - \alpha x \sin\alpha t)$.

Note that $\frac{\partial \gamma_t}{\partial x} = (\cos\alpha t, -\alpha\sin\alpha t)$, $\frac{\partial \gamma_t}{\partial p} = (\frac{1}{\alpha}\sin\alpha t, \cos\alpha t)$

$\frac{\partial \gamma_t}{\partial t} = (-\alpha x\sin\alpha t + p\cos\alpha t, -p\alpha\sin\alpha t - \alpha^2 x\cos\alpha t)$

so $\gamma_t(x,p)$ is smooth wrt x, p, t. Also,

$\gamma_s \circ \gamma_t(x,p)$

$= \gamma_s(x\cos\alpha t + \frac{p}{\alpha}\sin\alpha t,\ p\cos\alpha t - \alpha x\sin\alpha t)$

$= (x\cos\alpha t\cos\alpha s + \frac{p}{\alpha}\sin\alpha t\cos\alpha s + \frac{p}{\alpha}\cos\alpha t\sin\alpha s - x\sin\alpha t\sin\alpha s,$

$p\cos\alpha t\cos\alpha s - \alpha x\sin\alpha t\cos\alpha s - \alpha x\cos\alpha t\sin\alpha s - p\sin\alpha t\sin\alpha s)$

$= (x\cos\alpha(t+s) + \frac{p}{\alpha}\sin\alpha(t+s),\ p\cos\alpha(t+s) - \alpha x\sin\alpha(t+s))$

$= \gamma_{s+t}(x,p)$

Lastly, $\gamma_0(x,p) = (x,p)$, so $\{\gamma_t\}$ form a LOPGOLD for V_H

1e) Note that $\gamma_t \circ \gamma_s = \gamma_{t+s} = \gamma_{s+t} = \gamma_s \circ \gamma_t$ since this is a LOPGOLD, so the diffeomorphisms commute. Also, $\gamma_t \circ \gamma_s = \gamma_{t+s}$ is also a diffeomorphism so the set of diffeomorphisms in (d) is closed. γ_0 is the identity as shown in (id). Also, for $r, s, t \in I \subseteq \mathbb{R}$

$$\begin{aligned}
\gamma_r \circ (\gamma_s \circ \gamma_t) &= \gamma_r \circ \gamma_{s+t} \\
&= \gamma_{r+(s+t)} \\
&= \gamma_{(r+s)+t} \\
&= \gamma_{r+s} \circ \gamma_t \\
&= (\gamma_r \circ \gamma_s) \circ \gamma_t
\end{aligned}$$

so the function composition is associative.

Lastly, $\forall t \in I$, $\gamma_t \circ \gamma_{-t} = \gamma_{t-t} = \gamma_0 = \gamma_{-t+t} = \gamma_{-t} \circ \gamma_t$.

So, we have an abelian group.

2ai) Let G be the set of conformal diffeomorphisms on M. Then, for $F_1, F_2 \in G$, $F_1^* g = \Omega_1 \cdot g$, $F_2^* g = \Omega_2 \cdot g$, and

$$\begin{aligned}
(F_2 \cdot F_1)^* g &= F_1^* F_2^* g \\
&= F_1^* \Omega_2 \cdot g \\
&= \Omega_2 F_1^* \cdot g \quad \text{since } F_1^* \text{ is linear} \\
&= \Omega_2 \cdot \Omega_1 \cdot g
\end{aligned}$$

Note that $\Omega_2 \cdot \Omega_1$ is non-singular since neither Ω_2 nor Ω_1 is singular. So, $F_2 \cdot F_1 \in G$ and the set G is closed. Also, for $F_3 \in G$, $F_3^* g = \Omega_3 \cdot g$

$$\begin{aligned}
(F_3 \cdot (F_2 \cdot F_1))^* g &= (F_2 \cdot F_1)^* F_3^* g \\
&= F_1^* F_2^* \Omega_3 \cdot g \\
&= \Omega_3 \Omega_2 \Omega_1 \cdot g
\end{aligned}$$

$$\begin{aligned}
((F_3 \cdot F_2) \cdot F_1)^* g &= F_1^* (F_3 \cdot F_2)^* g \\
&= F_1^* F_2^* F_3^* g = \Omega_1 \Omega_2 \Omega_3 g = \text{LHS}
\end{aligned}$$

so the operation is associative. The identity map, $I(p) = p \ \forall p \in M$ is the identity for G because $I^* g = g$, so $\forall f \in G$, $(I \cdot F)^* g = F^* I^* g = F^* g$. Lastly, since F is a diffeomorphism, it is invertible, and

$$g = I^* g = (F F^{-1})^* g = F^{-1*} F g = F^{-1*} \Omega g = \Omega F^{-1*} g$$

So, $F^{-1*} g = \Omega^{-1} g$, where we can invert Ω because it is non singular. Thus, G is a group.

2aii) We need to know $F^* dx'^\mu = ?$

Let $F^* dx'^\mu = w_{\mu\lambda} dx^\lambda$

$$w_{\mu\lambda} = \langle F^* dx'^\mu, \frac{\partial}{\partial x^\lambda} \rangle$$

$$= \langle dx'^\mu, F_* \frac{\partial}{\partial x^\lambda} \rangle$$

$$F_* \frac{\partial f}{\partial x^\lambda}(x) = \frac{\partial}{\partial x^\lambda} f \circ F(x)$$

$$= \frac{\partial}{\partial x^\lambda} f(x'^1, \ldots, x'^n)$$

$$= \frac{\partial f}{\partial x'^\sigma} \frac{\partial x'^\sigma}{\partial x^\lambda}$$

So, $w_{\mu\lambda} = \langle dx'^\mu, \frac{\partial F^\sigma(x)}{\partial x^\lambda} \frac{\partial}{\partial x'^\sigma} \rangle$

$$= \frac{\partial F^\sigma(x)}{\partial x^\lambda} \delta^\mu_\sigma$$

$$= \frac{\partial F^\mu}{\partial x^\lambda}$$

$$F^* dx'^\mu = \frac{\partial F^\mu}{\partial x^\lambda} dx^\lambda$$

So, letting $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$,

$F^* g_{\alpha\beta}(x') dx'^\alpha \otimes dx'^\beta$

$= g_{\alpha\beta}(x') F^* dx'^\alpha \otimes F^* dx'^\beta$

$= g_{\alpha\beta}(x') \frac{\partial F^\alpha}{\partial x^\mu} dx^\mu \otimes \frac{\partial F^\beta}{\partial x^\nu} dx^\nu$

$= g_{\alpha\beta}(x') \frac{\partial F^\alpha}{\partial x^\mu} \frac{\partial F^\beta}{\partial x^\nu} dx^\mu \otimes dx^\nu$ since $\otimes$ is linear.

But RHS $= \Omega . g = \Omega g_{\mu\nu} dx^\mu \otimes dx^\nu$, so

$$g_{\alpha\beta}(x') \frac{\partial F^\alpha}{\partial x^\mu} \frac{\partial F^\beta}{\partial x^\nu} = \Omega(x) . g_{\mu\nu}$$

2biii) $\mathcal{L}_V g = \mathcal{L}_V (g_{\mu\nu} dx^\mu \otimes dx^\nu)$

$= \mathcal{L}_V(g_{\mu\nu}) dx^\mu \otimes dx^\nu + g_{\mu\nu} \mathcal{L}_V(dx^\mu) \otimes dx^\nu$

$+ g_{\mu\nu} dx^\mu \otimes \mathcal{L}_V(dx^\nu)$

Note that $\mathcal{L}_V(dx^\mu) = d(dx^\mu(V)) + \underbrace{d(dx^\mu)}_{=0}(V)$

$= d(V^\mu)$

$= \frac{\partial V^\mu}{\partial x^\lambda} dx^\lambda$

So, $\mathcal{L}_V g = \mathcal{L}_V(g_{\mu\nu}) dx^\mu \otimes dx^\nu + g_{\mu\nu} \frac{\partial V^\mu}{\partial x^\lambda} dx^\lambda \otimes dx^\nu$

$+ g_{\mu\nu} dx^\mu \otimes \frac{\partial V^\nu}{\partial x^\lambda} dx^\lambda$

$= \left(g_{\alpha\nu} \frac{\partial V^\alpha}{\partial x^\mu} + g_{\mu\alpha} \frac{\partial V^\alpha}{\partial x^\nu}\right) dx^\mu \otimes dx^\nu$ since g is constant.

So, $g_{\alpha\nu} \frac{\partial V^\alpha}{\partial x^\mu} + g_{\mu\alpha} \frac{\partial V^\alpha}{\partial x^\nu} = \Omega g_{\mu\nu}$, and note that $g_{\mu\nu} = g_{\nu\mu}$ since the metric tensor is symmetric.

2b(iv) $V^\alpha g_{\alpha\mu} = V_\mu$, $V^\alpha g_{\alpha\nu} = V_\nu$, so

$$\Omega g_{\mu\nu} = g_{\alpha\mu}\frac{\partial V^\alpha}{\partial x^\nu} + g_{\alpha\nu}\frac{\partial V^\alpha}{\partial x^\mu}$$

$$= \partial_\nu V_\mu + \partial_\mu V_\nu \quad (1)$$

$$\Omega \underbrace{g_{\mu\nu} g^{\mu\nu}}_{4 \text{ because } g = \operatorname{diag}(-1,1,1,1)} = g^{\nu\mu}\partial_\nu V_\mu + \partial_\mu V_\nu g^{\nu\mu}$$

$$4\Omega = \partial_\nu V^\nu + \partial_\mu V^\mu = 2\partial_\mu V^\mu$$

$\Omega = \frac{1}{2}\partial_\mu V^\mu$, so (1) becomes

$$\partial_\nu V_\mu + \partial_\mu V_\nu = \frac{1}{2}\partial_\lambda V^\lambda g_{\mu\nu}.$$

3a) Since θ is a one-form, $\theta = g\,df$ for some $f, g \in C^\infty(M)$.

So, $d\theta = dg \wedge df = dg \otimes df - df \otimes dg$.

LHS:

$d\theta(X, Y)$

$= dg(X)df(Y) - df(X)dg(Y)$

But $dg(X) = \langle \frac{\partial g}{\partial x^i} dx^i, X^i \frac{\partial}{\partial x^i}\rangle = \frac{\partial g}{\partial x^i}X^i = X^i\frac{\partial g}{\partial x^i} = X(g)$

so,

LHS $= d\theta(X,Y) = X(g)Y(f) - X(f)Y(g)$

Next, $X(\theta(Y)) = X\langle g\,df, Y\rangle$

$= X\, g.df(Y)$

$= X(g.Y(f))$

$= X(g)Y(f) + g\,X.Y(f)$

$= X(g)Y(f) + g.df(XY)$

$= X(g)Y(f) + \theta(XY)$

So, RHS $= X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y])$

$= X(g)Y(f) + \theta(XY) - Y(g)X(f) - \theta(YX)$

$- \theta(XY - YX)$

$= X(g)Y(f) - X(f)Y(g) + \theta(XY) - \theta(YX) - \theta(XY) + \theta(YX)$

$= X(g)Y(f) - X(f)Y(g)$

$=$ LHS

3b) Let $X = E_a, Y = E_b$, so

$$\theta([X,Y]) = \theta^c([E_a, E_b])$$
$$= \theta^c(C_{ab}^d E_d)$$
$$= C_{ab}^d \delta_d^c$$
$$= C_{ab}^c.$$

So, the expression in 3a becomes

$$d\theta^c(E_a, E_b)$$
$$= E_a \delta_b^c - E_b \delta_a^c - C_{ab}^c$$
$$= -C_{ab}^c \quad \text{assuming } c \neq a,\ c \neq b.$$

The expression in 3b leads to

$$d\theta^c(E_a, E_b) = -\tfrac{1}{2} C_{\mu\nu}^c \theta^\mu \wedge \theta^\nu (E_a, E_b)$$
$$= -\tfrac{1}{2} C_{\mu\nu}^c (\delta_a^\mu \delta_b^\nu - \delta_b^\mu \delta_a^\nu)$$
$$= -\tfrac{1}{2}(C_{ab}^c - C_{ba}^c)$$
$$= -C_{ab}^c \quad \text{since } C_{ab}^c = -C_{ba}^c$$

So the two expressions are equivalent.

3ci) $\langle \theta^1, E_1 \rangle = \langle \frac{1}{\beta_1} d\beta_1, \beta_1 \frac{\partial}{\partial \beta_1} \rangle = \frac{\beta_1}{\beta_1} = 1 = \delta_1^1$

$\langle \theta^1, E_2 \rangle = \langle \frac{1}{\beta_1} d\beta_1, \beta_1 \frac{\partial}{\partial \beta_2} \rangle = \langle d\beta_1, \frac{\partial}{\partial \beta_2} \rangle = 0 = \delta_2^1$

$\langle \theta^2, E_1 \rangle = \langle \frac{1}{\beta_1} d\beta_2, \beta_1 \frac{\partial}{\partial \beta_1} \rangle = \langle d\beta_2, \frac{\partial}{\partial \beta_1} \rangle = 0 = \delta_1^2$

$\langle \theta^2, E_2 \rangle = \langle \frac{1}{\beta_1} d\beta_2, \beta_1 \frac{\partial}{\partial \beta_2} \rangle = 1 = \delta_2^2$

so the duality conditions are satisfied.

3cii) We first calculate the structure constants. We know that $C_{11}^i = C_{22}^i = 0$ for $i = 1, 2$, since $[E_1, E_1] = [E_2, E_2] = 0$. For $f \in C^\infty(M)$,

$$[E_1, E_2]f = E_1 E_2 f - E_2 E_1 f$$
$$= \beta_1 \frac{\partial}{\partial \beta_1} \beta_1 \frac{\partial f}{\partial \beta_2} - \beta_1 \frac{\partial}{\partial \beta_2} \beta_1 \frac{\partial f}{\partial \beta_1}$$
$$= \beta_1 \frac{\partial f}{\partial \beta_2}$$
$$= E_2 f \implies C_{12}^1 = 0,\quad C_{12}^2 = 1$$

Evaluating the expression in (3b) for $c = 1$,

$$d\theta^1 + \tfrac{1}{2}(C_{12}^1 \theta^1 \wedge \theta^2 + C_{21}^1 \theta^2 \wedge \theta^1) \qquad C_{21}^1 = -C_{12}^1$$
$$= d\theta^1 + \underbrace{C_{12}^1}_{=0} \theta^1 \wedge \theta^2$$
$$= -\frac{1}{\beta_1^2} d\beta_1 \wedge d\beta_1 - \frac{1}{\beta_1^2} \underbrace{\frac{d\beta_1}{d\beta_2}}_{=0} d\beta_2 \wedge d\beta_1$$
$$= 0 \quad \text{since } d^2 = 0.$$

For $c=2$,

$$d\theta^2 + \frac{1}{2}\left(C_{12}^{\,2}\,\theta^1\wedge\theta^2 + C_{21}^{\,2}\,\theta^2\wedge\theta^1\right)$$

$$= d\theta^2 + C_{12}^{\,2}\,\theta^1\wedge\theta^2$$

$$= -\frac{1}{\beta_1^2}\,d\beta_1\wedge d\beta_2 + \frac{1}{\beta_1^2}\,d\beta^1\wedge d\beta^2$$

$$= 0.$$