PC4274

NATIONAL UNIVERSITY OF SINGAPORE

PC4274 - Mathematical Methods III

(Semester 2: AY2009-10)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **THREE** questions and comprises **FOUR** printed pages including this page.
2. Answer ALL **THREE (3)** questions.
3. All questions carry equal marks.
4. This is a CLOSED BOOK examination. Students are, however, allowed to bring into the examination hall, one A4-size "help" sheet with notes printed/written on both sides.

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(1) The symplectic manifold $\mathcal{M}$ associated with a particle that is constrained to move in one spatial dimension is characterized by the symplectic form

$$\omega = dx \wedge dp$$

where x denotes the spatial coordinate and p the momentum coordinate. For a system with Hamiltonian H, the Hamiltonian vector field V_H is defined through

$$\omega(V_H) = dH.$$

Given that the Hamiltonian of the system is

$$H = \frac{1}{2}\left(p^2 + \alpha^2 x^2\right)$$

where α is a constant, answer the following questions:

(a) Write down the equations of the integral curves associated with V_H.

(b) Solve the system of equations, given that at time $t = 0$, the position and the momentum of the particle is $x = x_0$ and $p = p_0$ respectively.

(c) Is V_H a complete vector field? Explain.

(d) Furnish the local one-parameter group of local diffeomorphisms associated with V_H.

(e) Show that the diffeomorphisms found in part (1d) constitute an abelian group.

(**2a**) A diffeomorphism $\mathcal{F}$ of a Riemannian manifold $(\mathcal{M}, g)$ into itself is conformal if

$$\mathcal{F}^* g = \Omega \cdot g$$

where Ω is a non-singular function on the manifold.

(i) Show that the set of conformal diffeomorphisms form a group.

(ii) In a coordinate chart, if $x^\mu \to x'^\mu = \mathcal{F}^\mu(x)$ is a conformal diffeomorphism, show that the following holds:

$$g_{\alpha\beta}(x') \frac{\partial \mathcal{F}^\alpha}{\partial x^\mu} \frac{\partial \mathcal{F}^\beta}{\partial x^\nu} = \Omega(x) \cdot g_{\mu\nu}(x).$$

(**2b**) A conformal Killing vector field V satisfies

$$\mathcal{L}_V g = \Omega \cdot g.$$

(iii) If g is the Minkowski metric, $g = \text{diag}(-1, 1, 1, 1)$, show that the components of V satisfy

$$g_{\alpha\mu} \frac{\partial V^\alpha}{\partial x^\nu} + g_{\alpha\nu} \frac{\partial V^\alpha}{\partial x^\mu} = \Omega g_{\mu\nu}.$$

(iv) By suitable contractions with the metric tensor, show that the expression in part (iii) reduces to

$$\partial_\mu V_\nu + \partial_\nu V_\mu = \frac{1}{2} \partial_\lambda V^\lambda g_{\mu\nu}$$

where $\partial_\mu \equiv \partial / \partial x^\mu$.

(3a) On a manifold $\mathcal{M}$, show that the following expression holds:

$$d\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y])$$

where θ is a one-form and X, Y are smooth vector fields.

(3b) If $\mathcal{M}$ is a group manifold and $\{\theta^a\}_{a=1,2,\ldots\dim(\mathcal{M})}$ is a set of one-forms that are dual to the left-invariant vector fields $\{E_a\}_{a=1,2,\ldots\dim(\mathcal{M})}$, in the sense

$$\langle \theta^a, E_b \rangle = \delta^a_b,$$

then show that the expression in part (3a) for $\theta = \theta^c$ is equivalent to

$$d\theta^c + \frac{1}{2}\sum_{a,b} C^c_{ab}\theta^a \wedge \theta^b = 0$$

Here C^c_{ab} are the structure constants of the Lie algebra $[E_a, E_b] = C^c_{ab}E_c$.

(3c) The closure relation $\Psi : G \times G \to G$ for the group G of collinear transformations of the real line, is given by

$$\Psi(\beta_1, \beta_2; \alpha_1, \alpha_2) = (\gamma_1, \gamma_2)$$

where

$$\gamma_1 = \beta_1\alpha_1, \qquad \gamma_2 = \beta_1\alpha_2 + \beta_2, \qquad \alpha_1, \beta_1 \neq 0.$$

The corresponding set of left-invariant vector fields $\{E_a\}_{a=1,2}$ and left-invariant one-forms $\{\theta^a\}_{a=1,2}$ are given by

$$E_1 = \beta_1\frac{\partial}{\partial\beta_1}, \qquad E_2 = \beta_1\frac{\partial}{\partial\beta_2}, \qquad \theta^1 = \frac{1}{\beta_1}d\beta_1, \qquad \theta^2 = \frac{1}{\beta_1}d\beta_2.$$

(i) Show that the duality conditions between the set of left-invariant vector fields and the forms are satisfied.

(ii) Show that the expression in part (3b), relating the one-forms, holds for the case considered here.

(KS)

****** END OF PAPER ******