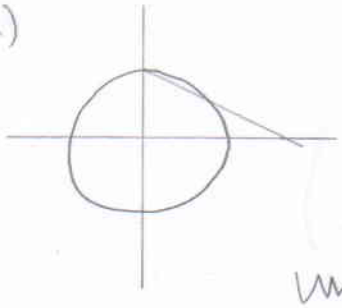
PC4374 2010-2011 Final

1a)



To do stereographic projection, we draw a line from the north pole to the point on the sphere. The line is described by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} x \\ y \\ z-1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

When it intersects the x-y plane at $(X, Y, 0)$,

$$0 = 1 + \lambda(z-1) \Rightarrow \lambda = \frac{1}{1-z} \Rightarrow X = \frac{x}{1-z}, \; Y = \frac{y}{1-z}$$

So, $\phi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$

For the ellipsoid, the north pole is at $(0,0,c)$, so the line is

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} + \lambda \begin{pmatrix} u \\ v \\ w-c \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

At the u-v plane, $0' = c + \lambda(w-c) \Rightarrow \lambda = \frac{c}{c-w}$

so, $u' = \frac{cu}{c-w}$, $v' = \frac{cv}{c-w} \Rightarrow \hat{\phi}(u,v,w) = \left(\frac{cu}{c-w}, \frac{cv}{c-w}\right)$

1b) $\overline{F}(X,Y) = \hat{\phi} \circ F \circ \phi^{-1}(X,Y)$

Now, we need to know $\phi^{-1}(X,Y)$.

① $X = \frac{x}{1-z}$, $Y = \frac{y}{1-z}$, where $z = \pm\sqrt{1-x^2-y^2}$

we need to solve for x, y, z in terms of X and Y. Substitute ① into the sphere equation:

$$X^2(1-2z+z^2) + Y^2(1-2z+z^2) + z^2 - 1 = 0$$

$$(X^2+Y^2+1)z^2 - 2(X^2+Y^2)z + X^2+Y^2-1 = 0$$

$$z = \frac{2(X^2+Y^2) \pm \sqrt{4(X^2+Y^2)^2 - 4(X^2+Y^2+1)(X^2+Y^2-1)}}{2(X^2+Y^2+1)}$$

$$= \frac{2(X^2+Y^2) \pm \sqrt{4(X^2+Y^2)^2 - 4(X^2+Y^2)^2 + 4}}{2(X^2+Y^2+1)}$$

$$= \frac{2(X^2+Y^2) \pm 2}{2(X^2+Y^2+1)}$$

$$= \frac{X^2+Y^2 \pm 1}{X^2+Y^2+1}$$

$$= \frac{X^2+Y^2-1}{X^2+Y^2+1}$$ (reject the "+" because that corresponds to the north pole).

Then, $x = X\left(1 - \frac{X^2+Y^2-1}{X^2+Y^2+1}\right) = \frac{2X}{X^2+Y^2+1}$

$y = \frac{2Y}{X^2+Y^2+1}$

so, $\phi^{-1}(X,Y) = \left(\frac{2X}{X^2+Y^2+1}, \frac{2Y}{X^2+Y^2+1}, \frac{X^2+Y^2-1}{X^2+Y^2+1}\right)$

$$F\circ\phi^{-1}(X,Y) = \left(\frac{2aX}{X^2+Y^2+1}, \frac{2bY}{X^2+Y^2+1}, \frac{c(X^2+Y^2-1)}{X^2+Y^2+1}\right)$$

$$\bar{F}(X,Y) = \hat{\phi}\circ F\circ\phi^{-1}(X,Y)$$

$$= \left(\frac{\frac{2acX}{X^2+Y^2+1}}{c-\frac{c(X^2+Y^2-1)}{X^2+Y^2+1}}, \frac{\frac{2bcY}{X^2+Y^2+1}}{c-\frac{c(X^2+Y^2-1)}{X^2+Y^2+1}}\right)$$

$$= \left(\frac{2acX}{2c}, \frac{2bcY}{2c}\right)$$

$$(u',v') = (aX, bY)$$

1c) $\bar{F}_*\frac{\partial}{\partial X}f(X,Y) = \frac{\partial}{\partial X} f\circ\bar{F}(X,Y)$

$$= \frac{\partial}{\partial X} f(u',v')$$

$$= \frac{\partial f}{\partial u'}\frac{\partial u'}{\partial X} + \frac{\partial f}{\partial v'}\frac{\partial v'}{\partial X}$$

$$\bar{F}_*\frac{\partial}{\partial X} = a\frac{\partial}{\partial u'}$$

Similarly, $\bar{F}_*\frac{\partial}{\partial Y} = \frac{\partial u'}{\partial Y}\frac{\partial}{\partial u'} + \frac{\partial v'}{\partial Y}\frac{\partial}{\partial v'} = b\frac{\partial}{\partial v'}$

where u', v' are the coordinates under $\hat{\phi}$ projection.

So $\bar{F}_*\bar{K} = X\bar{F}_*\frac{\partial}{\partial Y} - Y\bar{F}_*\frac{\partial}{\partial X}$

$$= \frac{u'}{a}b\frac{\partial}{\partial v'} - \frac{v'}{b}a\frac{\partial}{\partial u'}$$

$$= \frac{b}{a}u'\frac{\partial}{\partial v'} - \frac{a}{b}v'\frac{\partial}{\partial u'}$$

1d) $\frac{dY}{dt} = X, \quad \frac{dX}{dt} = -Y \Rightarrow \frac{d^2X}{dt^2} = -X, \quad \frac{d^2Y}{dt^2} = -Y$

So, $X(t) = A\cos t + B\sin t$

$Y(t) = C\cos t + D\sin t$

Letting $X(0)=X_0$, $Y(0)=Y_0$, $X(t) = X_0\cos t + B\sin t$

$Y(t) = Y_0\cos t + D\sin t$

$-Y_0 = \left.\frac{dX}{dt}\right|_{t=0} = B, \quad X_0 = \left.\frac{dY}{dt}\right|_{t=0} = D$, so

$X(t) = X_0\cos t - Y_0\sin t$

$Y(t) = Y_0\cos t + X_0\sin t$ gives the integral curve.

1e) The integral curve associated with $\hat{K}$ satisfy the equations

$$\frac{du'}{dt} = -\frac{a}{b}v', \quad \frac{dv'}{dt} = \frac{b}{a}u'$$

So, $\frac{d^2u'}{dt^2} = -u', \quad \frac{d^2v'}{dt^2} = -v'$

so, $u'(t) = u_0' \cos t + B \sin t$ where $u'(0) = u_0'$, $v'(0) = v_0'$.

$v'(t) = v_0' \cos t + D \sin t$.

$-\frac{a}{b} v_0' = \frac{du'}{dt}\Big|_{t=0} = B$, $\frac{b}{a} u_0' = \frac{dv'}{dt}\Big|_{t=0} = D$

The integral curves associated with $\tilde{K}$ have the following equations:

$u'(t) = u_0' \cos t - \frac{a}{b} v_0' \sin t$

$v'(t) = v_0' \cos t + \frac{b}{a} u_0' \sin t$.

Now, we map C under F, i.e find $\hat{C} = F \circ C$

$\hat{C}$: $u'(t) = aX_0 \cos t - aY_0 \sin t$

$v'(t) = bY_0 \cos t + bX_0 \sin t$.

Also, $u_0' = aX_0$, $v_0' = bY_0$, so

$\hat{C}$: $u'(t) = u_0' \cos t - \frac{a}{b} v_0' \sin t$

$v'(t) = v_0' \cos t + \frac{b}{a} u_0' \sin t$

so $\hat{C}$ is the integral curve associated with $\hat{K}$.

2a) $g = d\theta \otimes d\theta + \sin\theta\, d\phi \otimes \sin\theta\, d\phi$

so $e^1 = d\theta$, $e^2 = \sin\theta\, d\phi$.

and $de^1 = d^2\theta = 0$, $de^2 = \frac{\partial \sin\theta}{\partial\theta} d\theta \wedge d\phi = \cos\theta\, d\theta \wedge d\phi$

Also, the affine spin connection one-forms are antisymmetric with respect to a change in indices, i.e, $w^a_{\ b} = -w^b_{\ a}$, so

$w^\theta_{\ \theta} = -w^\theta_{\ \theta} \Rightarrow w^\theta_{\ \theta} = 0$. Similarly, $w^\phi_{\ \phi} = 0$. Let $e^1 = e^\theta$, $e^2 = e^\phi$

Now, $de^\theta + w^\theta_{\ \theta} \wedge e^\theta + w^\theta_{\ \phi} \wedge e^\phi = 0$.

$w^\theta_{\ \phi} \wedge e^\phi = 0$

$(w^\theta_{\ \phi})_\theta e^\theta \wedge e^\phi + (w^\theta_{\ \phi})_\phi e^\phi \wedge e^\phi = 0$

Then $(w^\theta_{\ \phi})_\theta = 0$ so $w^\theta_{\ \phi} = (w^\theta_{\ \phi})_\phi e^\phi$.

Also, $de^\phi + w^\phi_{\ \theta} \wedge e^\theta + \underbrace{w^\phi_{\ \phi}}_{=0} \wedge e^\phi = 0$

$\cos\theta\, d\theta \wedge d\phi + w^\phi_{\ \theta} \wedge e^\theta = 0$

$\cos\theta\, d\theta \wedge d\phi + w^\phi_{\ \theta} \wedge d\theta = 0$

$w^\phi_{\ \theta} \wedge d\theta = \cos\theta\, d\phi \wedge d\theta$

$\Rightarrow w^\phi_{\ \theta} = \cos\theta\, d\phi$, and $w^\theta_{\ \phi} = -\cos\theta\, d\phi$.

Now, we calculate the curvature 2-form.

$R^\theta_{\ \theta} = dw^\theta_{\ \theta} + w^\theta_{\ \theta} \wedge w^\theta_{\ \theta} + w^\theta_{\ \phi} \wedge \underbrace{w^\phi_{\ \theta}}_{=-w^\theta_{\ \phi}} = 0$.

similarly, $R^\phi_{\ \phi} = dw^\phi_{\ \phi} + w^\phi_{\ \theta} \wedge w^\theta_{\ \phi} + w^\phi_{\ \phi} \wedge w^\phi_{\ \phi} = 0$.

$R^\theta_{\ \phi} = dw^\theta_{\ \phi} + w^\theta_{\ \theta} \wedge w^\theta_{\ \phi} + w^\theta_{\ \phi} \wedge w^\phi_{\ \phi} = -\frac{\partial \cos\theta}{\partial\theta} d\theta \wedge d\phi = \sin\theta\, d\theta \wedge d\phi$.

$R^\phi_{\ \theta} = dw^\phi_{\ \theta} + w^\phi_{\ \theta} \wedge w^\theta_{\ \theta} + w^\phi_{\ \phi} \wedge w^\phi_{\ \theta} = \frac{\partial \cos\theta}{\partial\theta} d\theta \wedge d\phi = -\sin\theta\, d\theta \wedge d\phi$

2b) $e'^1 = \cos\alpha\, e^1 - \sin\alpha\, e^2$

$e'^2 = \sin\alpha\, e^1 + \cos\alpha\, e^2$

So, $g' = e'^1 \otimes e'^1 + e'^2 \otimes e'^2$

$= (\cos\alpha\, e^1 - \sin\alpha\, e^2) \otimes (\cos\alpha\, e^1 - \sin\alpha\, e^2)$

$+ (\sin\alpha\, e^1 + \cos\alpha\, e^2) \otimes (\sin\alpha\, e^1 + \cos\alpha\, e^2)$

$= \cos^2\alpha\, e^1 \otimes e^1 - \cos\alpha \sin\alpha\, e^1 \otimes e^2 - \sin\alpha\cos\alpha\, e^2 \otimes e^1$

$+ \sin^2\alpha\, e^2 \otimes e^2 + \sin^2\alpha\, e^1 \otimes e^1 + \sin\alpha\cos\alpha\, e^1 \otimes e^2$

$+ \cos\alpha\sin\alpha\, e^2 \otimes e^1 + \cos^2\alpha\, e^2 \otimes e^2.$

$= e^1 \otimes e^1 + e^2 \otimes e^2$

$= g.$

2c) $\omega^{\phi}_{\theta} = \cos\theta\, d\phi = \cot\theta\, e^2$

3a) $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R_z \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \sin\theta'\cos\phi' \\ \sin\theta'\sin\phi' \\ \cos\theta' \end{pmatrix} = \begin{pmatrix} \sin\theta\cos(\alpha+\phi) \\ \sin\theta\sin(\alpha+\phi) \\ \cos\theta \end{pmatrix}$$

So, $F_{R_z}(\theta, \phi) = (\theta, \alpha+\phi)$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R_x \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \sin\theta'\cos\phi' \\ \sin\theta'\sin\phi' \\ \cos\theta' \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi\cos\alpha - \sin\alpha\cos\theta \\ \sin\theta\sin\phi\sin\alpha + \cos\alpha\cos\theta \end{pmatrix}$$

We have, $\tan\phi' = \tan\phi\cos\alpha - \dfrac{\sin\alpha\cot\theta}{\cos\phi}$

$\cos\theta' = \sin\theta\sin\phi\sin\alpha + \cos\alpha\cos\theta$, so

$$F_{R_x}(\theta,\phi) = (\underbrace{\cos^{-1}(\sin\theta\sin\phi\sin\alpha + \cos\alpha\cos\theta)}_{\theta'}, \underbrace{\tan^{-1}\left(\tan\phi\cos\alpha - \frac{\sin\alpha\cot\theta}{\cos\phi}\right)}_{\phi'})$$

3b) $K_\alpha^i(x)\frac{\partial}{\partial x^i}$ are the Killing vectors in the coordinate chart.

$$K_z = \frac{\partial F^i(\alpha, \theta, \phi)}{\partial \theta}\bigg|_{\alpha=0} \frac{\partial}{\partial x^i}$$

$$= \underbrace{\frac{\partial\theta'}{\partial\alpha}\bigg|_{\alpha=0}}_{=0} \frac{\partial}{\partial\theta} + \underbrace{\frac{\partial\phi'}{\partial\alpha}}_{=1}\bigg|_{\alpha=0} \frac{\partial}{\partial\phi}$$

$$= \frac{\partial}{\partial\phi}$$

$$K_x = \frac{\partial\theta'}{\partial\alpha}\bigg|_{\alpha=0} \frac{\partial}{\partial\theta} + \frac{\partial\phi'}{\partial\alpha}\bigg|_{\alpha=0} \frac{\partial}{\partial\phi}$$

$$= -\frac{(\sin\theta\sin\phi\cos\alpha - \sin\alpha\cos\theta)}{\sqrt{1-(\sin\theta\sin\phi\sin\alpha + \cos\alpha\cos\theta)^2}}\bigg|_{\alpha=0} \frac{\partial}{\partial\theta}$$

$$+ \frac{-\tan\phi\sin\alpha - \frac{1}{\cos\phi}\cos\alpha\cot\theta}{1+\left(\tan\phi\cos\alpha - \frac{\sin\alpha\cot\theta}{\cos\phi}\right)^2}\bigg|_{\alpha=0} \frac{\partial}{\partial\phi}$$

$$= -\frac{\sin\theta\sin\phi}{\sqrt{1-\cos^2\theta}}\frac{\partial}{\partial\theta} - \frac{\frac{\cot\theta}{\cos\phi}}{1+\tan^2\phi}\frac{\partial}{\partial\phi}$$

$$= -\sin\phi \frac{\partial}{\partial\theta} - \frac{\cot\theta}{\cos\phi} \cos^2\phi \frac{\partial}{\partial\phi}$$

$$= -\sin\phi \frac{\partial}{\partial\theta} - \cot\theta\cos\phi \frac{\partial}{\partial\phi}.$$

3c) $\mathcal{L}_{K_z}(d\theta) = d(d\theta(K_z)) + (d\,d\theta)(K_z)$

$= d(0)$ because $\langle d\theta, \frac{\partial}{\partial\phi} \rangle = 0$

$= 0.$

$\mathcal{L}_{K_z}(d\phi) = d(d\phi(K_z)) + (d(d\phi))(K_z) = d(1) = 0$

$\mathcal{L}_{K_z}(\sin^2\theta) = K_z(\sin^2\theta) = \frac{\partial}{\partial\phi}\sin^2\theta = 0$

Thus, $\mathcal{L}_{K_z} g = (\mathcal{L}_{K_z} d\theta) \otimes d\theta + d\theta \otimes \mathcal{L}_{K_z}(d\theta)$

$+ \mathcal{L}_{K_z}(\sin^2\theta)\, d\phi \otimes d\phi + \sin^2\theta(\mathcal{L}_{K_z} d\phi) \otimes d\phi$

$+ \sin^2\theta\, d\phi \otimes \mathcal{L}_{K_z}(d\phi)$

$= 0.$

Now,

$L_{K_x} d\theta = d(d\theta(K_x)) + (d(d\theta))(K_x)$

$= d(-\sin\phi)$

$= -\cos\phi\, d\phi$

$\mathcal{L}_{K_x}(\sin^2\theta) = K_x(\sin^2\theta)$

$= -\sin\phi(2\sin\theta\cos\theta)$

$= -2\sin\phi\sin\theta\cos\theta.$

$\mathcal{L}_{K_x} d\phi = d(d\phi(K_x)) + (d(d\phi))(K_x)$

$= d(-\cot\theta\cos\phi)$

$= \operatorname{cosec}^2\theta\cos\phi\, d\theta + \cot\theta\sin\phi\, d\phi.$

So,

$\mathcal{L}_{K_x} g = -\cos\phi\, d\phi \otimes d\theta - \cos\phi\, d\theta \otimes d\phi$

$- 2\sin\phi\sin\theta\cos\theta\, d\phi \otimes d\phi$

$+ \cos\phi\, d\theta \otimes d\phi + \sin\theta\cos\theta\sin\phi\, d\phi \otimes d\phi.$

$+ \cos\phi\, d\phi \otimes d\theta + \sin\theta\cos\theta\sin\phi\, d\phi \otimes d\phi.$

$= 0.$