NATIONAL UNIVERSITY OF SINGAPORE

PC4274 MATHEMATICAL METHODS IN PHYSICS III

(Semester II: AY 2010-11)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **3** questions and comprises **4** printed pages including this page.
2. Answer **ALL THREE (3)** questions.
3. All questions carry equal marks.
4. Answers to the questions are to be written in the answer books.
5. This is a CLOSED BOOK examination.
6. One Help Sheet (A4 size, both sides) is allowed for this examination.

PC4274 - Mathematical Methods III

(**1**) Consider the 2-sphere, $\mathcal{M}$, defined by points $(x, y, z) \in \mathbf{R}^3$ that are constrained by the equation,

$$x^2 + y^2 + z^2 = 1.$$

$\mathcal{N}$ is another 2-dimensional manifold which is defined by points $(u, v, w) \in \mathbf{R}^3$ such that

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} + \frac{w^2}{c^2} = 1, \qquad a, b, c \in \mathbf{R}.$$

Points in $\mathcal{M}$ are mapped into $\mathcal{N}$ through the map $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ which is defined by

$$x \to u = a \cdot x, \qquad y \to v = b \cdot y, \qquad z \to w = c \cdot z.$$

A chart (U, ϕ) of $\mathcal{M}$ is given by considering the open set $U = \mathcal{M}\backslash\{0, 0, 1\}$ and stereographically mapping the points to the $x - y$ plane, *i.e.*,

$$\phi : (x, y, z) \in U \longrightarrow (X, Y) \in \mathbf{R}^2.$$

Similarly, one can also define a chart for $\mathcal{N}$, $(V, \widetilde{\phi})$, by stereographically projecting points $V = \mathcal{N}\backslash\{0, 0, c\}$ to the $u - v$ plane. In connection with this, answer the following questions:

(a) Evaluate the the projective coordinates of $\mathcal{M}$ and $\mathcal{N}$ relative to the charts ϕ and $\widetilde{\phi}$ respectively.

(b) Provide an explicit realization of the induced map $\overline{\mathcal{F}} : \phi(U) \to \widetilde{\phi}(V)$.

(c) A vector field K on $\mathcal{M}$ is realized by

$$\overline{K} = \phi_* K = X\frac{\partial}{\partial Y} - Y\frac{\partial}{\partial X}$$

in its local chart. Evaluate the push-forward of $\overline{K}$ relative to $\overline{\mathcal{F}}$.

(d) Evaluate the integral curve $\mathcal{C}$, corresponding to $\overline{K}$.

(e) Show that $\widetilde{\mathcal{C}} = \overline{\mathcal{F}} \circ \mathcal{C}$ is the integral curve associated with $\widetilde{K} = \overline{\mathcal{F}}_* \overline{K}$.

(2) On an N-dimensional Riemannian manifold $\mathcal{M}$ with metric tensor

$$g = e^1 \otimes e^1 + e^2 \otimes e^2 + \cdots + e^N \otimes e^N,$$

where $\{e^i\}_{i=1,2,\ldots,N}$ constitutes a basis of 1-forms, one defines the affine spin connection one-form $\omega^a{}_b$ through

$$de^a + \omega^a{}_b \wedge e^b = 0$$

and the curvature two-form $R^a{}_b$ as

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b.$$

In connection with this, consider the 2-sphere (S^2) with a metric tensor given by

$$g = d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi$$

where (θ, ϕ) are the usual spherical coordinates of the sphere.

(a) Evaluate $\omega^a{}_b$ and $R^a{}_b$ for the 2-sphere.

(b) Show that the transformation $e^a \longrightarrow e'^a = \Phi^a{}_b e^b$, where Φ is a matrix of the form

$$\Phi = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

leaves the metric invariant. Here $\alpha \equiv \alpha(\theta, \phi)$ is an arbitrary smooth function of θ and ϕ.

(c) Show that under the transformation in part (b), the affine spin connection one-form transforms as

$$\omega^a{}_b \to \omega'^a_b = \omega^a{}_b + d\alpha.$$

(d) Show that the components of the curvature 2-form remain invariant under the transformation in part (b), *i.e.*,

$$R^a{}_b \to R'^a_b = R^a{}_b.$$

(**3**) A transformation of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathbf{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $\mathbf{R}$ is a 3×3 orthogonal matrix, induces a transformation ($\mathcal{F}_R$) of S^2 onto itself. Here S^2 is regarded as a surface in $\mathbf{R}^3$. For instance, in spherical coordinates, a point characterized by (θ, ϕ), transforms to (θ', ϕ') under the induced transformation:

$$\mathcal{F}_R : (\theta, \phi) \rightarrow (\theta', \phi').$$

In connection with this, consider transformations of the form,

$$\mathbf{R}_z = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{R}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}.$$

(a) Furnish the induced transformations $\mathcal{F}_{R_z}$ and $\mathcal{F}_{R_x}$ associated with $\mathbf{R}_z$ and $\mathbf{R}_x$ respectively.

(b) Show that the Killing vectors associated with $\mathcal{F}_{R_z}$ and $\mathcal{F}_{R_x}$ are given by

$$\begin{aligned} K_z &= \frac{\partial}{\partial\phi} \\ \text{and} \qquad K_x &= -\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi} \end{aligned}$$

respectively.

(c) Show that $\mathcal{L}_{K_z} g = 0 = \mathcal{L}_{K_x} g$, where

$$g = d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi$$

is the metric tensor on S^2.

(KS)

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