P(4)711 2011-2012 final

1a) $X = \sinh\chi\cos\varphi$

$Y = \sinh\chi\sin\varphi$

$Z = \cosh\chi$

Note that $X^2+Y^2 = \sinh^2\chi \Rightarrow \chi = \sinh^{-1}\sqrt{X^2+Y^2}$

$\frac{Y}{X} = \tan\varphi \Rightarrow \varphi = \tan^{-1}\frac{Y}{X}$

So, $\bar{F}(X,Y) = (\sinh^{-1}\sqrt{X^2+Y^2}, \tan^{-1}\frac{Y}{X})$

1b) We first need to find $\bar{F}^* d\chi$ and $\bar{F}^* d\varphi$. We shall use the transformation rule for covariant vectors.

$(\bar{F}^* d\chi)_X = \frac{\partial\chi}{\partial X} = \frac{-2X}{\sqrt{1-X^2-Y^2}}$

$(\bar{F}^* d\chi)_Y = \frac{\partial\chi}{\partial Y} = \frac{-2Y}{\sqrt{1-X^2-Y^2}}$ $\Rightarrow \bar{F}^* d\chi = -\frac{2X}{\sqrt{1-X^2-Y^2}}dX - \frac{2Y}{\sqrt{1-X^2-Y^2}}dY$

$(\bar{F}^* d\varphi)_X = \frac{\partial\varphi}{\partial X} = -\frac{1}{1+\frac{Y^2}{X^2}}\frac{Y}{X^2} = -\frac{Y}{X^2+Y^2}$

$(\bar{F}^* d\varphi)_Y = \frac{\partial\varphi}{\partial Y} = \frac{1}{1+\frac{Y^2}{X^2}}\frac{1}{X} = \frac{X}{X^2+Y^2} \Rightarrow \bar{F}^* d\varphi = -\frac{Y}{X^2+Y^2}dX + \frac{X}{X^2+Y^2}dY$

We can also obtain $\bar{F}^* d\chi$ and $\bar{F}^* d\varphi$ by letting

$\bar{F}^* d\chi = \alpha_X dX + \alpha_Y dY$, $\bar{F}^* d\varphi = \beta_X dX + \beta_Y dY$

Then,

$\alpha_X = \langle \bar{F}^* d\chi, \frac{\partial}{\partial X}\rangle = \langle d\chi, \bar{F}_* \frac{\partial}{\partial X}\rangle$

Now, we need $\bar{F}_* \frac{\partial}{\partial X}$ and $\bar{F}_* \frac{\partial}{\partial Y}$

$\bar{F}_* \frac{\partial}{\partial X} f(X,Y) = \frac{\partial}{\partial X} f\, \bar{F}(X,Y)$

$= \frac{\partial}{\partial X} f(\chi,\varphi)$

$= \frac{\partial f}{\partial\chi}\frac{\partial\chi}{\partial X} + \frac{\partial f}{\partial\varphi}\frac{\partial\varphi}{\partial X}$

$\bar{F}_* \frac{\partial}{\partial X} = \frac{-2X}{\sqrt{1-X^2-Y^2}}\frac{\partial}{\partial\chi} - \frac{1}{1+\frac{Y^2}{X^2}}\frac{Y}{X^2}\frac{\partial}{\partial\varphi}$

$\bar{F}_* \frac{\partial}{\partial X} = -\frac{2X}{\sqrt{1-X^2-Y^2}}\frac{\partial}{\partial\chi} - \frac{Y}{X^2+Y^2}\frac{\partial}{\partial\varphi}$

$\bar{F}_* \frac{\partial}{\partial Y} = \frac{\partial\chi}{\partial Y}\frac{\partial}{\partial\chi} + \frac{\partial\varphi}{\partial Y}\frac{\partial}{\partial\varphi} = -\frac{2Y}{\sqrt{1-X^2-Y^2}}\frac{\partial}{\partial\chi} + \frac{1}{1+\frac{Y^2}{X^2}}\frac{1}{X}\frac{\partial}{\partial\varphi}$

$= -\frac{2Y}{\sqrt{1-X^2-Y^2}}\frac{\partial}{\partial\chi} + \frac{X}{X^2+Y^2}\frac{\partial}{\partial\varphi}$

So, $\alpha_X = \langle d\chi, -\frac{2X}{\sqrt{1-X^2-Y^2}}d\chi - \frac{Y}{X^2+Y^2}d\varphi\rangle = -\frac{2X}{\sqrt{1-X^2-Y^2}}$

$\alpha_Y = \langle \bar{F}^* d\chi, \frac{\partial}{\partial Y}\rangle = \langle d\chi, \bar{F}_* \frac{\partial}{\partial Y}\rangle = -\frac{2Y}{\sqrt{1-X^2-Y^2}}$

$\beta_X = \langle \bar{F}^* d\varphi, \frac{\partial}{\partial X}\rangle = \langle d\varphi, \bar{F}_* \frac{\partial}{\partial X}\rangle = -\frac{Y}{X^2+Y^2}$

$\beta_Y = \langle d\varphi, \bar{F}_* \frac{\partial}{\partial Y}\rangle = \frac{X}{X^2+Y^2}$

So, $\bar{F}^* dX = \frac{-2X}{\sqrt{1-X^2-Y^2}} dX - \frac{2Y}{\sqrt{1-X^2-Y^2}} dY$

$\bar{F}^* d\varphi = -\frac{Y}{X^2+Y^2} dX + \frac{X}{X^2+Y^2} dY$

which is what we got earlier. Now,

$$\bar{g}_2 = \bar{F}^* \bar{g}_1$$
$$= \bar{F}^*(dX \otimes dX + \sinh^2 X \, d\varphi \otimes d\varphi)$$
$$= \bar{F}^* dX \otimes \bar{F}^* dX + \sinh^2 X \, \bar{F}^* d\varphi \otimes \bar{F}^* d\varphi$$
$$= \left(-\frac{2X}{\sqrt{1-X^2-Y^2}} dX - \frac{2Y}{\sqrt{1-X^2-Y^2}} dY\right) \otimes \left(-\frac{2X}{\sqrt{1-X^2-Y^2}} dX - \frac{2Y}{\sqrt{1-X^2-Y^2}} dY\right)$$
$$+ (x^2+y^2)\left(-\frac{Y}{X^2+Y^2} dX + \frac{X}{X^2+Y^2} dY\right) \otimes \left(-\frac{Y}{X^2+Y^2} dX + \frac{X}{X^2+Y^2} dY\right)$$
$$= \left(\frac{4X^2}{1-X^2-Y^2} + \frac{Y^2}{X^2+Y^2}\right) dX \otimes dX + \left(\frac{4XY}{1-X^2-Y^2} - \frac{XY}{X^2+Y^2}\right) dX \otimes dY$$
$$+ \left(\frac{4YX}{1-X^2-Y^2} - \frac{XY}{X^2+Y^2}\right) dY \otimes dX + \left(\frac{4Y^2}{1-X^2-Y^2} + \frac{X^2}{X^2+Y^2}\right) dY \otimes dY$$

1b) Since neither of these basis are orthonormal with respect to the metric, we will need to find $\sqrt{|\bar{g}_1|}$, $\sqrt{|\bar{g}_2|}$, but first, we write down their matrix expression.

$(\bar{g}_1)_{XX} = \bar{g}_1\left(\frac{\partial}{\partial X}, \frac{\partial}{\partial X}\right) = 1$, $(\bar{g}_1)_{\varphi\varphi} = \sinh^2 X$. $(\bar{g}_1)_{X\varphi} = (\bar{g}_1)_{\varphi X} = 0$.

So $\bar{g}_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 X \end{pmatrix}$ and $|\bar{g}_1| = \sinh^2 X$ so $\sqrt{|\bar{g}_1|} = \sinh X$

Similarly, $\bar{g}_2 = \begin{pmatrix} \frac{4X^2}{1-X^2-Y^2} + \frac{Y^2}{X^2+Y^2} & \frac{4XY}{1-X^2-Y^2} - \frac{XY}{X^2+Y^2} \\ \frac{4YX}{1-X^2-Y^2} - \frac{XY}{X^2+Y^2} & \frac{4Y^2}{1-X^2-Y^2} + \frac{X^2}{X^2+Y^2} \end{pmatrix}$

$$|\bar{g}_2| = \frac{16X^2Y^2}{1-X^2-Y^2} + \frac{4X^4}{(1-X^2-Y^2)(X^2+Y^2)} + \frac{4Y^4}{(1-X^2-Y^2)(X^2+Y^2)} + \frac{X^2Y^2}{(X^2+Y^2)^2}$$
$$- \frac{16X^2Y^2}{(1-X^2-Y^2)} + \frac{4X^2Y^2}{(1-X^2-Y^2)(X^2+Y^2)} + \frac{4X^2Y^2}{(1-X^2-Y^2)(X^2+Y^2)} - \frac{X^2Y^2}{(X^2+Y^2)^2}$$
$$= \frac{4(X^4+2X^2Y^2+Y^4)}{(1-X^2-Y^2)(X^2+Y^2)}$$
$$= \frac{4(X^2+Y^2)}{1-X^2-Y^2} \Rightarrow \sqrt{|\bar{g}_2|} = 2\sqrt{\frac{X^2+Y^2}{1-X^2-Y^2}}$$

So, $\bar{\mu}_1 = \sqrt{|\bar{g}_1|}\, dX \wedge d\varphi = \sinh X \, dX \wedge d\varphi$.

$\bar{\mu}_2 = 2\sqrt{\frac{X^2+Y^2}{1-X^2-Y^2}}\, dX \wedge dY$

1c) $\bar{F}^*\bar{\mu}_1 = \bar{F}^*(\sinh X \, dX \wedge d\varphi)$

$$= \sinh X \, (\bar{F}^* dX) \wedge (\bar{F}^* d\varphi)$$
$$= \sqrt{x^2+y^2}\left(-\frac{2X}{\sqrt{1-X^2-Y^2}} dX - \frac{2Y}{\sqrt{1-X^2-Y^2}} dY\right) \wedge \left(-\frac{Y}{X^2+Y^2} dX + \frac{X}{X^2+Y^2} dY\right)$$
$$= \sqrt{x^2+y^2}\left(-\frac{2X^2}{(X^2+Y^2)\sqrt{1-X^2-Y^2}} dX \wedge dY + \frac{2Y^2}{(X^2+Y^2)\sqrt{1-X^2-Y^2}} dY \wedge dX\right)$$

since $dX \wedge dX = 0 = dY \wedge dY$

$$= \sqrt{x^2+y^2}\left(-\frac{2(x^2+y^2)}{(x^2+y^2)\sqrt{1-x^2-y^2}}\, dx\wedge dy\right)$$

$$= -2\sqrt{\frac{x^2+y^2}{1-x^2-y^2}}\, dx\wedge dy$$

$$= -\bar{\mu}_2$$

To get rid of the minus sign, we should have defined $\bar{\mu}_1 = \sinh\chi\, d\varphi\wedge d\chi$ instead of $\bar{\mu}_1 = \sinh\chi\, d\chi\wedge d\varphi$

2a) We first need to find F_χ^{-1}

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \end{pmatrix} \xrightarrow{F_\chi^{-1}} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 \\ -\sinh\chi & \cosh\chi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \end{pmatrix}$$

$$x^0 = x'^0\cosh\chi - x'^1\sinh\chi$$
$$x^1 = -x'^0\sinh\chi + x'^1\cosh\chi$$
$$x^2 = x'^2$$

Let $F_\chi^{-1*}dx^0 = a_{00}\,dx'^0 + a_{01}\,dx'^1 + a_{02}\,dx'^2$

$F_\chi^{-1*}dx^1 = a_{10}\,dx'^0 + a_{11}\,dx'^1 + a_{12}\,dx'^2$

$F_\chi^{-1*}dx^2 = a_{20}\,dx'^0 + a_{21}\,dx'^1 + a_{22}\,dx'^2$

We need $F_{\chi*}^{-1}\frac{\partial}{\partial x'^0}$, $F_{\chi*}^{-1}\frac{\partial}{\partial x'^1}$, $F_{\chi*}^{-1}\frac{\partial}{\partial x'^2}$

$$F_{\chi*}^{-1}\frac{\partial}{\partial x'^0} f(x'^0, x'^1, x'^2) = \frac{\partial}{\partial x'^0} f(x^0, x^1, x^2)$$
$$= \frac{\partial f}{\partial x^0}\frac{\partial x^0}{\partial x'^0} + \frac{\partial f}{\partial x^1}\frac{\partial x^1}{\partial x'^0} + \frac{\partial f}{\partial x^2}\frac{\partial x^2}{\partial x'^0}$$

$$F_{\chi*}^{-1}\frac{\partial}{\partial x'^0} = \cosh\chi\frac{\partial}{\partial x^0} - \sinh\chi\frac{\partial}{\partial x^1}$$

$$F_{\chi*}^{-1}\frac{\partial}{\partial x'^1} = \frac{\partial x^0}{\partial x'^1}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial x'^1}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial x'^1}\frac{\partial}{\partial x^2} = -\sinh\chi\frac{\partial}{\partial x^0} + \cosh\chi\frac{\partial}{\partial x^1}$$

$$F_{\chi*}^{-1}\frac{\partial}{\partial x'^2} = \frac{\partial x^0}{\partial x'^2}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial x'^2}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial x'^2}\frac{\partial}{\partial x^2} = \frac{\partial}{\partial x^2}$$

$a_{00} = \langle F_\chi^{-1*}dx^0, \frac{\partial}{\partial x'^0}\rangle = \langle dx^0, F_{\chi*}^{-1}\frac{\partial}{\partial x'^0}\rangle = \cosh\chi$

$a_{01} = \langle F_\chi^{-1*}dx^0, \frac{\partial}{\partial x'^1}\rangle = \langle dx^0, F_{\chi*}^{-1}\frac{\partial}{\partial x'^1}\rangle = -\sinh\chi$

$a_{02} = \langle dx^0, F_{\chi*}^{-1}\frac{\partial}{\partial x'^2}\rangle = 0 \Rightarrow F_\chi^{-1*}dx^0 = \cosh\chi\, dx'^0 - \sinh\chi\, dx'^1$

$a_{10} = \langle F_\chi^{-1*}dx^1, \frac{\partial}{\partial x'^0}\rangle = \langle dx^1, F_{\chi*}^{-1}\frac{\partial}{\partial x'^0}\rangle = -\sinh\chi$

$a_{11} = \langle dx^1, F_{\chi*}^{-1}\frac{\partial}{\partial x'^1}\rangle = \cosh\chi$

$a_{12} = \langle dx^1, F_{\chi*}^{-1}\frac{\partial}{\partial x'^2}\rangle = 0 \Rightarrow F_\chi^{-1*}dx^1 = -\sinh\chi\, dx'^0 + \cosh\chi\, dx'^1$

$a_{20} = \langle dx^2, F_{\chi*}^{-1}\frac{\partial}{\partial x'^0}\rangle = 0$, $a_{21} = 0$

$a_{22} = \langle dx^2, F_{\chi*}^{-1}\frac{\partial}{\partial x'^2}\rangle = 1$, so $F_\chi^{-1*}dx^2 = dx'^2$.

So,

$$F' = F_\chi^{-1*} F$$

$$= -E_1 (F_\chi^{-1*} dx^0) \wedge (F_\chi^{-1*} dx^1) - E_2 (F_\chi^{-1*} dx^0) \wedge (F_\chi^{-1*} dx^2)$$
$$+ B (F_\chi^{-1*} dx^1) \wedge (F_\chi^{-1*} dx^2)$$

$$= -E_1 (\cosh^2\chi \, dx'^0 \wedge dx'^1 + \sinh^2\chi \, dx'^1 \wedge dx'^0)$$
$$- E_2 (\cosh\chi \, dx'^0 \wedge dx'^2 - \sinh\chi \, dx'^1 \wedge dx'^2)$$
$$+ B (-\sinh\chi \, dx'^0 \wedge dx'^2 + \cosh\chi \, dx'^1 \wedge dx'^2)$$

$$= - E_1 \, dx'^0 \wedge dx'^1 - (E_2 \cosh\chi + B \sinh\chi) \, dx'^0 \wedge dx'^2$$
$$+ (E_2 \sinh\chi + B \cosh\chi) \, dx'^1 \wedge dx'^2.$$

But $F' = - E_1' dx'^0 \wedge dx'^1 - E_2' dx'^0 \wedge dx'^2 + B' dx'^1 \wedge dx'^2$

So, $E_1' = E_1$

$E_2' = E_2 \cosh\chi + B \sinh\chi$

$B' = E_2 \sinh\chi + B \cosh\chi$

2 bi) We first show this for 1-forms, $w = w_0 dx^0 + w_1 dx^1 + w_2 dx^2$

$$d \circ F_\chi^{-1*} w$$
$$= d(w_0 F_\chi^{-1*} dx^0 + w_1 F_\chi^{-1*} dx^1 + w_2 F_\chi^{-1*} dx^2)$$
$$= d(w_0 (\cosh\chi \, dx'^0 - \sinh\chi \, dx'^1) + w_1 (-\sinh\chi \, dx'^0 + \cosh\chi \, dx'^1) + w_2 dx'^2)$$
$$= d((w_0 \cosh\chi - w_1 \sinh\chi) dx'^0 + (w_1 \cosh\chi - w_0 \sinh\chi) dx'^1 + w_2 dx'^2)$$
$$= \left(\frac{\partial w_0}{\partial x'^1} \cosh\chi - \frac{\partial w_1}{\partial x'^1} \sinh\chi\right) dx'^1 \wedge dx'^0$$
$$+ \left(\frac{\partial w_0}{\partial x'^2} \cosh\chi - \frac{\partial w_1}{\partial x'^2} \sinh\chi\right) dx'^2 \wedge dx'^0$$
$$+ \left(\frac{\partial w_1}{\partial x'^0} \cosh\chi - \frac{\partial w_0}{\partial x'^0} \sinh\chi\right) dx'^0 \wedge dx'^1$$
$$+ \left(\frac{\partial w_1}{\partial x'^2} \cosh\chi - \frac{\partial w_0}{\partial x'^2} \sinh\chi\right) dx'^2 \wedge dx'^1 + \frac{\partial w_2}{\partial x'^0} dx'^0 \wedge dx'^2$$
$$+ \frac{\partial w_2}{\partial x'^1} dx'^1 \wedge dx'^2$$
$$= \left(\frac{\partial w_1}{\partial x'^0} \cosh\chi - \frac{\partial w_0}{\partial x'^0} \sinh\chi - \frac{\partial w_0}{\partial x'^1} \cosh\chi + \frac{\partial w_1}{\partial x'^1} \sinh\chi\right) dx'^0 \wedge dx'^1$$
$$+ \left(\frac{\partial w_1}{\partial x'^2} \sinh\chi - \frac{\partial w_0}{\partial x'^2} \cosh\chi + \frac{\partial w_2}{\partial x'^0}\right) dx'^0 \wedge dx'^2$$
$$+ \left(\frac{\partial w_0}{\partial x'^2} \sinh\chi - \frac{\partial w_1}{\partial x'^2} \cosh\chi + \frac{\partial w_2}{\partial x'^1}\right) dx'^1 \wedge dx'^2.$$

RHS:

$$F_\chi^{-1*} \circ d(w_0 dx^0 + w_1 dx^1 + w_2 dx^2)$$
$$= F_\chi^{-1*} \left(\frac{\partial w_0}{\partial x^1} dx^1 \wedge dx^0 + \frac{\partial w_0}{\partial x^2} dx^2 \wedge dx^0 + \frac{\partial w_1}{\partial x^0} dx^0 \wedge dx^1 \right.$$
$$\left. + \frac{\partial w_1}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial w_2}{\partial x^0} dx^0 \wedge dx^2 + \frac{\partial w_2}{\partial x^1} dx^1 \wedge dx^2 \right)$$

$= F_x^{-1*}\left(\left(\frac{\partial w_1}{\partial x^0} - \frac{\partial w_0}{\partial x^1}\right) dx^0\wedge dx^1 + \left(\frac{\partial w_2}{\partial x^0} - \frac{\partial w_0}{\partial x^2}\right) dx^0 \wedge dx^2 + \left(\frac{\partial w_2}{\partial x^1} - \frac{\partial w_1}{\partial x^2}\right) dx^1\wedge dx^2\right)$

$= \left(\frac{\partial w_1}{\partial x^0} - \frac{\partial w_0}{\partial x^1}\right) dx'^0\wedge dx'^1$

$+ \left(\frac{\partial w_2}{\partial x^0} - \frac{\partial w_0}{\partial x^2}\right)\left(\cosh\chi\, dx'^0\wedge dx'^2 - \sinh\chi\, dx'^1\wedge dx'^2\right)$

$+ \left(\frac{\partial w_2}{\partial x^1} - \frac{\partial w_1}{\partial x^2}\right)\left(-\sinh\chi\, dx'^0\wedge dx'^2 + \cosh\chi\, dx'^1\wedge dx'^2\right)$

$= \left(\frac{\partial w_1}{\partial x^0} - \frac{\partial w_0}{\partial x^1}\right) dx'^0\wedge dx'^1$

$+ \left(\frac{\partial w_2}{\partial x^0}\cosh\chi - \frac{\partial w_0}{\partial x^2}\cosh\chi - \frac{\partial w_2}{\partial x^1}\sinh\chi + \frac{\partial w_1}{\partial x^2}\sinh\chi\right) dx'^0\wedge dx'^2$

$+ \left(\cosh\chi\frac{\partial w_2}{\partial x^1} - \frac{\partial w_1}{\partial x^2}\cosh\chi - \frac{\partial w_2}{\partial x^0}\sinh\chi + \frac{\partial w_0}{\partial x^2}\sinh\chi\right) dx'^1\wedge dx'^2$

But $\frac{\partial}{\partial x^0} = \frac{\partial x^{0'}}{\partial x^0}\frac{\partial}{\partial x^{0'}} + \frac{\partial x^{1'}}{\partial x^0}\frac{\partial}{\partial x^{1'}} + \frac{\partial x^{2'}}{\partial x^0}\frac{\partial}{\partial x^{2'}} = \cosh\chi\frac{\partial}{\partial x^{0'}} + \sinh\chi\frac{\partial}{\partial x^{1'}}$

$\frac{\partial}{\partial x^1} = \frac{\partial x^{0'}}{\partial x^1}\frac{\partial}{\partial x^{0'}} + \frac{\partial x^{1'}}{\partial x^1}\frac{\partial}{\partial x^{1'}} + \frac{\partial x^{2'}}{\partial x^1}\frac{\partial}{\partial x^{2'}} = \sinh\chi\frac{\partial}{\partial x^{0'}} + \cosh\chi\frac{\partial}{\partial x^{1'}}$

$\frac{\partial}{\partial x^{2'}} = \frac{\partial}{\partial x^2}$

So,

RHS

$= F_x^{-1*}\, dw$

$= \left(\cosh\chi\frac{\partial w_1}{\partial x'^0} + \sinh\chi\frac{\partial w_1}{\partial x'^1} - \sinh\chi\frac{\partial w_0}{\partial x'^0} - \cosh\chi\frac{\partial w_0}{\partial x'^1}\right) dx'^0\wedge dx'^1$

$+ \left(\cosh^2\chi\frac{\partial w_2}{\partial x'^0} + \cosh\chi\sinh\chi\frac{\partial w_2}{\partial x'^1} - \frac{\partial w_0}{\partial x'^2}\cosh\chi - \sinh^2\chi\frac{\partial w_2}{\partial x'^0}\right.$
$\left. - \sinh\chi\cosh\chi\frac{\partial w_2}{\partial x'^1} + \frac{\partial w_1}{\partial x'^2}\sinh\chi\right) dx'^0\wedge dx'^2$

$+ \left(\cosh\chi\sinh\chi\frac{\partial w_2}{\partial x'^0} + \cosh^2\chi\frac{\partial w_2}{\partial x'^1} - \frac{\partial w_1}{\partial x'^2}\cosh\chi - \sinh\chi\cosh\chi\frac{\partial w_2}{\partial x'^0}\right.$
$\left. - \sinh^2\chi\frac{\partial w_2}{\partial x'^1} + \sinh\chi\frac{\partial w_0}{\partial x'^2}\right) dx'^1\wedge dx'^2$

$= \left(\cosh\chi\frac{\partial w_1}{\partial x'^0} + \sinh\chi\frac{\partial w_1}{\partial x'^1} - \sinh\chi\frac{\partial w_0}{\partial x'^0} - \cosh\chi\frac{\partial w_0}{\partial x'^1}\right) dx'^0\wedge dx'^1$

$+ \left(\frac{\partial w_2}{\partial x'^0} - \frac{\partial w_0}{\partial x'^2}\cosh\chi + \frac{\partial w_1}{\partial x'^2}\sinh\chi\right) dx'^0\wedge dx'^2$

$+ \left(\frac{\partial w_2}{\partial x'^1} - \frac{\partial w_1}{\partial x'^2}\cosh\chi + \sinh\chi\frac{\partial w_0}{\partial x'^2}\right) dx'^1\wedge dx'^2$

$=$ LHS

Now we look at two-forms, $w = w_{01}\, dx^0\wedge dx^1 + w_{02}\, dx^0\wedge dx^2 + w_{12}\, dx^1\wedge dx^2$

LHS

$= d F_x^{-1*} w$

$= d\big(w_{01}(dx'^0\wedge dx'^1) + w_{02}(\cosh\chi\, dx'^0\wedge dx'^2 - \sinh\chi\, dx'^1\wedge dx'^2)$
$+ w_{12}(-\sinh\chi\, dx'^0\wedge dx'^2 + \cosh\chi\, dx'^1\wedge dx'^2)\big)$

$= d\big(w_{01}\, dx'^0\wedge dx'^1 + (w_{02}\cosh\chi - w_{12}\sinh\chi)\, dx'^0\wedge dx'^2$
$+ (w_{12}\cosh\chi - w_{02}\sinh\chi)\, dx'^1\wedge dx'^2\big)$

$= \left(\frac{\partial w_{01}}{\partial x'^2} - \frac{\partial w_{02}}{\partial x'^1}\cosh\chi + \frac{\partial w_{12}}{\partial x'^1}\sinh\chi + \frac{\partial w_{12}}{\partial x'^0}\cosh\chi - \frac{\partial w_{02}}{\partial x'^0}\sinh\chi\right) dx'^0\wedge dx'^1\wedge dx'^2$

RHS

$= F_x^{-1*} \circ d w$

$= F_x^{-1*}\left(\frac{\partial w_{01}}{\partial x^2} - \frac{\partial w_{02}}{\partial x^1} + \frac{\partial w_{12}}{\partial x^0}\right) dx^0 \wedge dx^1 \wedge dx^2$

$= \left(\frac{\partial w_{01}}{\partial x'^2} - \frac{\partial w_{02}}{\partial x'^0} \sinh\chi - \cosh\chi \frac{\partial w_{02}}{\partial x'^1} + \cosh\chi \frac{\partial w_{12}}{\partial x'^0} + \sinh\chi \frac{\partial w_{12}}{\partial x'^1}\right)$

$(\cosh\chi\, dx'^0 - \sinh\chi\, dx'^1) \wedge (-\sinh\chi\, dx'^0 + \cosh\chi\, dx'^1) \wedge (dx'^2)$

$= \left(\frac{\partial w_{01}}{\partial x'^2} - \frac{\partial w_{02}}{\partial x'^0} \sinh\chi - \cosh\chi \frac{\partial w_{02}}{\partial x'^1} + \cosh \frac{\partial w_{12}}{\partial x'^0} + \sinh\chi \frac{\partial w_{12}}{\partial x'^1}\right) dx'^0 \wedge dx'^1 \wedge dx'^2$

$= LHS.$

2bii) Letting $w = w_{01}\, dx^0 \wedge dx^1 + w_{02}\, dx^0 \wedge dx^2 + w_{12}\, dx^1 \wedge dx^2$

$* F_x^{-1*} w$

$= *\big(w_{01}\, dx'^0 \wedge dx'^1 + (w_{02} \cosh\chi - w_{12} \sinh\chi)\, dx'^0 \wedge dx'^2$

$+ (w_{12} \cosh\chi - w_{02} \sinh\chi)\, dx'^1 \wedge dx'^2 \big)$

Let us calculate the $*$ of basis two forms.

$*(dx'^0 \wedge dx'^1) = \frac{1}{1!} g^{0 j_1} g^{1 j_2} \varepsilon^{0\,1\,2}_{j_1 j_2 j_3} dx'^{j_3} = -dx'^2$

$*(dx'^0 \wedge dx'^2) = \frac{1}{1!} g^{0 j_1} g^{2 j_2} \varepsilon^{0\,1\,2}_{j_1 j_2 j_3} dx'^{j_3} = +dx'^1$

$*(dx'^1 \wedge dx'^2) = g^{1 j_1} g^{2 j_2} \varepsilon^{0\,1\,2}_{j_1 j_2 j_3} dx'^{j_3} = dx'^0$

where we assumed that $g' = -dx'^0 \otimes dx'^0 + dx'^1 \otimes dx'^1 + dx'^2 \otimes dx'^2$

and so $\sqrt{|g'|} = 1$. So,

$LHS = -w_{01} dx'^2 + (w_{02} \cosh\chi - w_{12} \sinh\chi)\, dx'^1$

$+ (w_{12} \cosh\chi - w_{02} \sinh\chi)\, dx'^0$

Now we calculate the RHS. The $*$ of the basis $\{dx^i\}_{i=0,1,2}$ is the same as that for $\{dx'^i\}_{i=0,1,2}$ since we assumed that g is invariant under Lorentz boost.

RHS

$= F_x^{-1*} \circ * w$

$= F_x^{-1*}\big(-w_{01}\, dx^2 + w_{02}\, dx^1 + w_{12}\, dx^0 \big)$

$= -w_{01}\, dx'^2 + w_{02}(-\sinh\chi\, dx'^0 + \cosh\chi\, dx'^1) + w_{12}(\cosh\chi\, dx'^0 - \sinh\chi\, dx'^1)$

$= -w_{01}\, dx'^2 + (w_{02} \cosh\chi - \sinh\chi\, w_{12})\, dx'^1 + (w_{12} \cosh\chi - w_{02} \sinh\chi)\, dx'^0$

2c) $dF' = d \circ F_x^{-1*} F = F_x^{-1*} \circ dF = 0$

$d * F' = d * F_x^{-1*} F = d F_x^{-1*} * F = F_x^{-1*} d * F = 0$

3a)

$$0 = \mathcal{L}_K \omega$$
$$= d(\omega(K)) + \underbrace{(d\omega)}_{=0}(K)$$
$$= d\left((dq^i \otimes dp^i - dp^i \otimes dq^i)\left(\alpha^i \frac{\partial}{\partial q^i} + \beta^i \frac{\partial}{\partial p^i} \right)\right)$$
$$= d\left(\alpha^i \delta^i_j \, dp^j - \beta^i \delta^i_j \, dq^j \right)$$
$$= d(\alpha^i dp^i - \beta^i dq^i)$$

Then, $\alpha^i dp^i - \beta^i dq^i = dH$ for some $H \in C^\infty(M)$, since $d^2 = 0$.

$$\alpha^i dp^i - \beta^i dq^i = \frac{\partial H}{\partial p^i} dp^i + \frac{\partial H}{\partial q^i} dq^i \Rightarrow \alpha^i = \frac{\partial H}{\partial p^i}, \ \beta^i = -\frac{\partial H}{\partial q^i}$$

b) Then the integral curves of K are

$$\frac{dq^i}{dt} = \alpha^i = \frac{\partial L}{\partial p^i}, \quad \frac{dp^i}{dt} = \beta^i = -\frac{\partial L}{\partial q^i}$$

For $f \in C^\infty(M)$, $\omega(X_f) = df$, letting $X_f = a^i \frac{\partial}{\partial q^i} + b^i \frac{\partial}{\partial p^i}$

$$a^i dp^i - b^i dq^i = \frac{\partial f}{\partial p^i} dp^i + \frac{\partial f}{\partial q^i} dq^i, \text{ so}$$

$$X_f = \frac{\partial f}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p^i}$$

So, $\{q^i, L\} = \omega(X_{q^i}, X_L)$

$$= (dq^i \otimes dp^i - dp^i \otimes dq^i)\left(-\frac{\partial}{\partial p^i}, \frac{\partial L}{\partial p^i}\frac{\partial}{\partial q^i} - \frac{\partial L}{\partial q^i}\frac{\partial}{\partial p^i}\right)$$
$$= \frac{\partial L}{\partial p^i} = \frac{dq^i}{dt}$$

$$\{p^i, L\} = (dq^i \otimes dp^i - dp^i \otimes dq^i)\left(\frac{\partial}{\partial q^i}, \frac{\partial L}{\partial p^i}\frac{\partial}{\partial q^i} - \frac{\partial L}{\partial q^i}\frac{\partial}{\partial p^i}\right)$$
$$= -\frac{\partial L}{\partial q^i} = \frac{dp^i}{dt}.$$