NATIONAL UNIVERSITY OF SINGAPORE

PC4274 MATHEMATICAL METHODS IN PHYSICS III

(Semester II: AY 2011-12)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **3** questions and comprises **4** printed pages including this page.
2. Answer **ALL THREE (3)** questions.
3. All questions carry equal marks.
4. Answers to the questions are to be written in the answer books.
5. This is a CLOSED BOOK examination.
6. One Help Sheet (A4 size, both sides) is allowed for this examination.

<u>**PC4274 - Mathematical Methods III**</u>

(**1**) Let $(\mathcal{M}, g)$ be a Riemannian manifold in $\mathbf{R}^3$ that satisfy

$$z^2 - x^2 - y^2 = 1 \qquad z > 0.$$

One can write down a chart ϕ_1 that maps an open neighborhood $U_1 \subset \mathcal{M}$:

$$\phi_1 : (x, y, z) \in U_1 \rightarrow (\chi, \varphi) \in \mathbf{R}^2$$

where

$$\begin{aligned} x &= \sinh\chi\cos\varphi \\ y &= \sinh\chi\sin\varphi \\ z &= \cosh\chi. \end{aligned}$$

Yet another chart (U_2, ϕ_2) can be defined as

$$\phi_2 : (x, y, z) \in U_2 \rightarrow (X, Y) \in \mathbf{R}^2$$

where

$$X = x, \qquad Y = y.$$

In the chart (U_1, ϕ_1), the local expression for the metric tensor g on $\mathcal{M}$ takes the form of,

$$\overline{g}_1 = \phi_1^{-1*} g = d\chi \otimes d\chi + \sinh^2 \chi d\varphi \otimes d\varphi.$$

In connection with this, answer the following questions:

(a) Evaluate the map $\overline{\mathcal{F}} : \phi_2(U_1 \cap U_2) \rightarrow \phi_1(U_1 \cap U_2)$ where

$$\overline{\mathcal{F}} : (X, Y) \rightarrow (\chi, \varphi)$$

and compute $\overline{g}_2 = \overline{\mathcal{F}}^* \overline{g}_1$, the metric tensor in chart (U_2, ϕ_2).

(b) Furnish the volume forms $\overline{\mu}_1$ and $\overline{\mu}_2$ in the charts (U_1, ϕ_1) and (U_2, ϕ_2) respectively.

(c) Show that the two volume forms are related by $\overline{\mathcal{F}}^* \overline{\mu}_1 = \overline{\mu}_2$.

(**2**) Given a three-dimensional space-time $\mathcal{M}$ with metric tensor

$$g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2$$

where x^0 denotes the temporal coordinate and x^1, x^2, the two spatial coordinates. On this manifold one defines an electromagnetic field tensor through

$$F = -E_1 dx^0 \wedge dx^1 - E_2 dx^0 \wedge dx^2 + B dx^1 \wedge dx^2.$$

Here E_1 and E_2 denotes the electric field components in the two spatial dimensions and B is the magnetic field. Note that the magnetic field has only one component. Now, suppose we perform a Lorentz-boost in the x^1 direction, $\mathcal{F}_\chi : \mathcal{M} \to \mathcal{M}$, where

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} \xrightarrow{\mathcal{F}_\chi} \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \end{pmatrix} = \begin{pmatrix} \cosh\chi & \sinh\chi & 0 \\ \sinh\chi & \cosh\chi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix}.$$

In connection with this, answer the following questions.

(a) By evaluating the pullback of F, $F' = \mathcal{F}_\chi^{-1*} F$, show that the electric field components and the magnetic field transform as

$$\begin{aligned} E_1 \to E_1' &= E_1; \\ E_2 \to E_2' &= E_2 \cosh\chi + B \sinh\chi; \\ B \to B' &= E_2 \sinh\chi + B \cosh\chi. \end{aligned}$$

(b) Show that following operations are equivalent:

(i) On the space of one-forms **and** two-forms,

$$d \circ \mathcal{F}_\chi^{-1*} = \mathcal{F}_\chi^{-1*} \circ d.$$

(ii) On the space of two-forms,

$$* \circ \mathcal{F}_\chi^{-1*} = \mathcal{F}_\chi^{-1*} \circ *.$$

where d in (i) denotes the exterior derivative and $*$ in (ii) denotes the hodge-star operator.
(Hint: You may assume that the metric remains invariant under the Lorentz boost.)

(c) Hence or otherwise, show that if F in the unboosted frame satisfies source-free Maxwell-type equations,

$$dF = 0, \qquad\qquad d * F = 0,$$

then the transformed field tensor F' also satisfies

$$dF' = 0, \qquad\qquad d * F' = 0.$$

(**3**) On a symplectic manifold $\mathcal{M}$, of N particles with cooordinates $(q^1, q^2, ..., q^{3N}, p^1, p^2, ..., p^{3N})$, one has the symplectic 2-form,

$$\omega = \sum_{i=1}^{3N} dq^i \wedge dp^i.$$

(a) For a vector field of the form

$$K = \sum_{i=1}^{3N} \alpha^i(p,q)\frac{\partial}{\partial q^i} + \beta^i(p,q)\frac{\partial}{\partial q^i},$$

what conditions must one impose on the α^i and β^i for $\mathcal{L}_K\omega = 0$?

(b) If the conditions of part (a) are met then show that the integral curves associated with K satisfy

$$\frac{dq^i}{dt} = \{q^i, L\}$$
$$\frac{dp^i}{dt} = \{p^i, L\}.$$

Here $\{\cdot,\cdot\}$ denotes the Poisson bracket and the zero-form L satisfies

$$\omega(K) = dL.$$

(KS)

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