PC4274 (AY2013/2014 sem 2)
Suggested solutions

Question 1

1a. Since $\bar{X}$ and $\bar{Y}$ are vector fields, they can be expressed as $\bar{X} = X^j \frac{\partial}{\partial x^j}$ and $\bar{Y} = Y^i \frac{\partial}{\partial x^i}$ in a coordinate basis. Since $\left[\bar{X}, \bar{Y}\right] = \bar{X}\bar{Y} - \bar{Y}\bar{X}$, we evaluate $\bar{X}\bar{Y}$ first:

$$\begin{aligned}\bar{X}\bar{Y} &= \left(X^j \frac{\partial}{\partial x^j}\right)\left(Y^i \frac{\partial}{\partial x^i}\right) \\ &= X^j \left(\frac{\partial}{\partial x^j} Y^i\right) \frac{\partial}{\partial x^i} + X^j Y^i \left(\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\right)\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\bar{Y}\bar{X} &= Y^j \left(\frac{\partial}{\partial x^j} X^i\right) \frac{\partial}{\partial x^i} + Y^j X^i \left(\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\right) \\ &= Y^j \left(\frac{\partial}{\partial x^j} X^i\right) \frac{\partial}{\partial x^i} + Y^j X^i \left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\right), \text{ assuming partial derivatives commute} \\ &= Y^j \left(\frac{\partial}{\partial x^j} X^i\right) \frac{\partial}{\partial x^i} + X^j Y^i \left(\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\right), \text{ relabelling summmation indices}\end{aligned}$$

Taking the difference, the latter terms in the two expressions cancel, giving us

$$\begin{aligned}\left[\bar{X}, \bar{Y}\right] = \bar{X}\bar{Y} - \bar{Y}\bar{X} &= X^j \left(\frac{\partial}{\partial x^j} Y^i\right) \frac{\partial}{\partial x^i} - Y^j \left(\frac{\partial}{\partial x^j} X^i\right) \frac{\partial}{\partial x^i} \\ &= \left(X^j \frac{\partial}{\partial x^j} Y^i - Y^j \frac{\partial}{\partial x^j} X^i\right) \frac{\partial}{\partial x^i}\end{aligned}$$

hence it is a vector field, with components

$$\left[\bar{X}, \bar{Y}\right]^i = X^j \frac{\partial}{\partial x^j} Y^i - Y^j \frac{\partial}{\partial x^j} X^i$$

1b. Since $\tilde{\alpha}$ is a 1-form, its exterior derivative is a 2-form, with components

$$\begin{aligned}(\tilde{\mathrm{d}}\tilde{\alpha})_{ij} &= (1+1)\partial_{[i}\alpha_{j]} \\ &= 2\left(\frac{1}{2!}\left(\partial_i \alpha_j - \partial_j \alpha_i\right)\right) \\ &= \frac{\partial \alpha_j}{\partial x_i} - \frac{\partial \alpha_i}{\partial x_j}\end{aligned}$$

1c. Using the result from part (b),

$$\begin{aligned}
\tilde{\mathrm{d}}\tilde{\alpha}\left(\bar{X},\bar{Y}\right) &= (\tilde{\mathrm{d}}\tilde{\alpha})_{ij}X^iY^j \\
&= \left(\frac{\partial \alpha_j}{\partial x_i} - \frac{\partial \alpha_i}{\partial x_j}\right)X^iY^j \\
&= X^i\frac{\partial \alpha_j}{\partial x_i}Y^j - Y^j\frac{\partial \alpha_i}{\partial x_j}X^i
\end{aligned}$$

We note that $\frac{\partial}{\partial x_i}\left(\alpha_j Y^j\right) = \frac{\partial \alpha_j}{\partial x_i}Y^j + \alpha_j\frac{\partial Y^j}{\partial x_i}$ by product rule, and so $\frac{\partial \alpha_j}{\partial x_i}Y^j = \frac{\partial}{\partial x_i}\left(\alpha_j Y^j\right) - \alpha_j\frac{\partial Y^j}{\partial x_i}$. Similarly, we have $\frac{\partial \alpha_i}{\partial x_j}X^i = \frac{\partial}{\partial x_j}\left(\alpha_i X^i\right) - \alpha_i\frac{\partial X^i}{\partial x_j}$. Substituting into the above gives

$$\begin{aligned}
\tilde{\mathrm{d}}\tilde{\alpha}\left(\bar{X},\bar{Y}\right) &= X^i\frac{\partial \alpha_j}{\partial x_i}Y^j - Y^j\frac{\partial \alpha_i}{\partial x_j}X^i \\
&= X^i\left(\frac{\partial}{\partial x_i}\left(\alpha_j Y^j\right) - \alpha_j\frac{\partial Y^j}{\partial x_i}\right) - Y^j\left(\frac{\partial}{\partial x_j}\left(\alpha_i X^i\right) - \alpha_i\frac{\partial X^i}{\partial x_j}\right) \\
&= X^i\frac{\partial}{\partial x_i}\left(\alpha_j Y^j\right) - Y^j\frac{\partial}{\partial x_j}\left(\alpha_i X^i\right) - X^i\alpha_j\frac{\partial Y^j}{\partial x_i} + Y^j\alpha_i\frac{\partial X^i}{\partial x_j} \\
&= \bar{X}\left(\alpha_j Y^j\right) - \bar{Y}\left(\alpha_i X^i\right) - X^i\alpha_j\frac{\partial Y^j}{\partial x_i} + Y^i\alpha_j\frac{\partial X^j}{\partial x_i}, \text{ relabelling indices on last term} \\
&= \bar{X}\tilde{\alpha}(\bar{Y}) - \bar{Y}\tilde{\alpha}(\bar{X}) - \alpha_j\left(X^i\frac{\partial Y^j}{\partial x_i} - Y^i\frac{\partial X^j}{\partial x_i}\right) \\
&= \bar{X}\tilde{\alpha}(\bar{Y}) - \bar{Y}\tilde{\alpha}(\bar{X}) - \alpha_j\left[\bar{X},\bar{Y}\right]^j, \text{ by the result of part (a)} \\
&= \bar{X}\tilde{\alpha}(\bar{Y}) - \bar{Y}\tilde{\alpha}(\bar{X}) - \tilde{\alpha}\left(\left[\bar{X},\bar{Y}\right]\right)
\end{aligned}$$

If $\tilde{\alpha}$ is exact, then by definition $\tilde{\alpha} = \tilde{\mathrm{d}}f$ for some scalar field f (since $\tilde{\alpha}$ is a 1-form). We hence have $\tilde{\alpha} = \tilde{\mathrm{d}}f = \frac{\partial f}{\partial x^i}\tilde{\mathrm{d}}x^i$ in a coordinate basis, and the right-hand side of the above equation then becomes

$$\begin{aligned}
\bar{X}\tilde{\alpha}(\bar{Y}) - \bar{Y}\tilde{\alpha}(\bar{X}) - \tilde{\alpha}\left(\left[\bar{X},\bar{Y}\right]\right) &= \bar{X}\frac{\partial f}{\partial x^i}\tilde{\mathrm{d}}x^i(\bar{Y}) - \bar{Y}\frac{\partial f}{\partial x^i}\tilde{\mathrm{d}}x^i(\bar{X}) - \frac{\partial f}{\partial x^i}\tilde{\mathrm{d}}x^i\left(\left[\bar{X},\bar{Y}\right]\right) \\
&= \bar{X}\frac{\partial f}{\partial x^i}Y^i - \bar{Y}\frac{\partial f}{\partial x^i}X^i - \frac{\partial f}{\partial x^i}\left[\bar{X},\bar{Y}\right]^i \\
&= \left(X^j\frac{\partial}{\partial x^j}\right)\left(\frac{\partial f}{\partial x^i}Y^i\right) - \left(Y^j\frac{\partial}{\partial x^j}\right)\left(\frac{\partial f}{\partial x^i}X^i\right) \\
&\quad - \frac{\partial f}{\partial x^i}\left(X^j\frac{\partial}{\partial x^j}Y^i - Y^j\frac{\partial}{\partial x^j}X^i\right), \text{ by the result of part (a)} \\
&= X^j\frac{\partial^2 f}{\partial x^j\partial x^i}Y^i + \cancel{X^j\frac{\partial f}{\partial x^i}\frac{\partial Y^i}{\partial x^j}} - Y^j\frac{\partial^2 f}{\partial x^j\partial x^i}X^i - \cancel{Y^j\frac{\partial f}{\partial x^i}\frac{\partial X^i}{\partial x^j}} \\
&\quad - \cancel{\frac{\partial f}{\partial x^i}X^j\frac{\partial Y^i}{\partial x^j}} + \cancel{\frac{\partial f}{\partial x^i}Y^j\frac{\partial X^i}{\partial x^j}} \\
&= X^j\frac{\partial^2 f}{\partial x^j\partial x^i}Y^i - Y^j\frac{\partial^2 f}{\partial x^j\partial x^i}X^i \\
&= X^j\frac{\partial^2 f}{\partial x^j\partial x^i}Y^i - Y^i\frac{\partial^2 f}{\partial x^i\partial x^j}X^j, \text{ relabelling summation indices} \\
&= 0, \text{ assuming partial derivatives commute}
\end{aligned}$$

Question 2

2a. One explicit coordinate basis would be $\left\{\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x, \tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y, \tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}z, \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y, \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z, \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z\right\}$.

The vector space has dimension ${}^4C_2 = 6$.

Since the manifold is 4-dimensional, 2-forms map to 2-forms $(4-2=2)$. Hence the vector space of 2-forms maps to itself under the dual map. To verify that the mapping is surjective (a.k.a. onto), we shall show that the above set of six basis 2-forms maps bijectively to itself (up to sign differences) under the dual map. As given in the hint, the dual mapping applied to 2-forms gives

$$*(\tilde{\mathrm{d}}x^a \wedge \tilde{\mathrm{d}}x^b) = \frac{1}{(4-2)!}\epsilon^{ab}{}_{cd}\,\tilde{\mathrm{d}}x^c \wedge \tilde{\mathrm{d}}x^d = \frac{1}{2}\epsilon^{ab}{}_{cd}\,\tilde{\mathrm{d}}x^c \wedge \tilde{\mathrm{d}}x^d$$

To figure out the value of $\epsilon^{ab}{}_{cd}$, we note that (assuming Minkowski space in standard coordinates with the $-+++$ sign convention, $g^{\mu\nu} = \eta^{\mu\nu}$) we have $\epsilon^{ab}{}_{cd} = \eta^{a\alpha}\eta^{b\beta}\epsilon_{\alpha\beta cd}$. Since $\eta^{a\alpha} = 0$ unless $a = \alpha$, all the terms in the summation over α, β on the RHS are zero except for the term with $\alpha = a$, $\beta = b$ (which may also be zero, if $a = b$). We can hence write

$$\epsilon^{ab}{}_{cd} = \eta^{aa}\eta^{bb}\epsilon_{abcd} \quad (\underline{\text{no sum}} \text{ over } a, b)$$

Therefore, $\epsilon^{ab}{}_{cd}$ and ϵ_{abcd} differ only by a sign, given by $\eta^{aa}\eta^{bb}$. Since $\eta^{00} = -1$ and $\eta^{11} = \eta^{22} = \eta^{33} = +1$, we see that $\epsilon^{ab}{}_{cd} = \epsilon_{abcd}$ if neither of a, b is 0, and $\epsilon^{ab}{}_{cd} = -\epsilon_{abcd}$ if either of a, b is 0. (If any of a, b, c, d are equal to each other, the expression evaluates to zero.)

It is convenient to now list the sign of ϵ_{abcd} and thus $\epsilon^{ab}{}_{cd}$ for some permutations we will use:

$\epsilon_{0123} = +1$	$\epsilon_{0213} = -1$	$\epsilon_{0321} = -1$	$\epsilon_{1203} = +1$	$\epsilon_{1320} = +1$	$\epsilon_{2310} = -1$
$\epsilon^{01}{}_{23} = -1$	$\epsilon^{02}{}_{13} = +1$	$\epsilon^{03}{}_{21} = +1$	$\epsilon^{12}{}_{03} = +1$	$\epsilon^{13}{}_{20} = +1$	$\epsilon^{23}{}_{10} = -1$

Using these values, we can hence write

$$\begin{aligned}
*(\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x) = *(\tilde{\mathrm{d}}x^0 \wedge \tilde{\mathrm{d}}x^1) &= \frac{1}{2}\epsilon^{01}{}_{cd}\,\tilde{\mathrm{d}}x^c \wedge \tilde{\mathrm{d}}x^d \\
&= \frac{1}{2}(-\tilde{\mathrm{d}}x^2 \wedge \tilde{\mathrm{d}}x^3 + \tilde{\mathrm{d}}x^3 \wedge \tilde{\mathrm{d}}x^2) \\
&= -\tilde{\mathrm{d}}x^2 \wedge \tilde{\mathrm{d}}x^3 \quad \text{since } \tilde{\mathrm{d}}x^3 \wedge \tilde{\mathrm{d}}x^2 = -\tilde{\mathrm{d}}x^2 \wedge \tilde{\mathrm{d}}x^3 \text{ by antisymmetry} \\
&= -\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z
\end{aligned}$$

and similarly we compute

$$\begin{aligned}
&*(\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y) = \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z \\
&*(\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}z) = -\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \\
&*(\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y) = \tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}z \\
&*(\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z) = -\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y \\
&*(\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z) = \tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x
\end{aligned}$$

Hence, the mapping of the vector space of 2-forms to itself under the dual map is surjective.

2bi. Since $\tilde{F}$ is a 2-form, $*\tilde{F}$ is also a 2-form, as described in part (a). It is easier to work with the $F^{\mu\nu}$ form instead of the $F_{\mu\nu}$ form given:

$$
\begin{aligned}
F^{\mu\nu} = g^{\mu\lambda}g^{\nu\sigma}F_{\lambda\sigma} &= g^{\mu\lambda}F_{\lambda\sigma}g^{\sigma\nu} \\
&= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}
\end{aligned}
$$

With these components, we can write $\tilde{F}$ as a sum of coordinate basis elements, then as a sum of 2-form basis elements:

$$
\begin{aligned}
\tilde{F} &= F_{\mu\nu}\,\tilde{\mathrm{d}}x^\mu \otimes \tilde{\mathrm{d}}x^\nu \\
&= E^1\tilde{\mathrm{d}}t \otimes \tilde{\mathrm{d}}x + E^2\tilde{\mathrm{d}}t \otimes \tilde{\mathrm{d}}y + E^3\tilde{\mathrm{d}}t \otimes \tilde{\mathrm{d}}z - E^1\tilde{\mathrm{d}}x \otimes \tilde{\mathrm{d}}t + B^3\tilde{\mathrm{d}}x \otimes \tilde{\mathrm{d}}y - B^2\tilde{\mathrm{d}}x \otimes \tilde{\mathrm{d}}z + (\text{etc.}) \\
&= E^1(\tilde{\mathrm{d}}t \otimes \tilde{\mathrm{d}}x - \tilde{\mathrm{d}}x \otimes \tilde{\mathrm{d}}t) + E^2(\tilde{\mathrm{d}}t \otimes \tilde{\mathrm{d}}y - \tilde{\mathrm{d}}t \otimes \tilde{\mathrm{d}}y) + (\text{etc.}) \\
&= E^1\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x + E^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y + E^3\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}z + B^3\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y - B^2\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z + B^1\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z
\end{aligned}
$$

Using the results of part (a), we can then find $*\tilde{F}$:

$$
\begin{aligned}
*\tilde{F} &= *(E^1\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x + E^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y + E^3\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}z + B^3\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y - B^2\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z + B^1\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z) \\
&= -E^1\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z + E^2\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z - E^3\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y + B^3\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}z + B^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y + B^1\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \\
&= B^1\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x + B^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y + B^3\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}z - E^3\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y + E^2\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z - E^1\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z
\end{aligned}
$$

It can hence be seen that the effect of the dual map on the electric and magnetic fields is that the resulting electric field takes the value of the original magnetic field, and the resulting magnetic field takes the value of the negative of the original electric field ($\boldsymbol{E}_f = \boldsymbol{B}_i, \boldsymbol{B}_f = -\boldsymbol{E}_i$).

2bii. By using the above expressions for $\tilde{F}$ and $*\tilde{F}$, we can compute $\tilde{F} \wedge *\tilde{F}$ by expanding the wedge product:

$$
\begin{aligned}
\tilde{F} \wedge *\tilde{F} = {} & (E^1\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x + E^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y + E^3\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}z + B^3\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y - B^2\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z + B^1\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z) \\
& \wedge (B^1\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x + B^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y + B^3\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}z - E^3\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y + E^2\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z - E^1\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z) \\
= {} & -(E^1)^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z + (E^2)^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z - (E^3)^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}z \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \\
& + (B^3)^2\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}z - (B^2)^2\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z \wedge \tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y + (B^1)^2\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z \wedge \tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x, \\
& \text{noting that all terms of the form } \tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y \text{ and so on are zero} \\
= {} & -(E^1)^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z - (E^2)^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z - (E^3)^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z \\
& + (B^3)^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z + (B^2)^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z + (B^1)^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z, \\
& \text{using the even/oddness of the permutations computed in part (a)} \\
= {} & (|\boldsymbol{B}|^2 - |\boldsymbol{E}|^2)\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z
\end{aligned}
$$

Therefore, $\displaystyle\int \tilde{F} \wedge *\tilde{F} = \int (|\boldsymbol{B}|^2 - |\boldsymbol{E}|^2)\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z = \int (|\boldsymbol{B}|^2 - |\boldsymbol{E}|^2)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$

Similarly for $\tilde{F} \wedge \tilde{F}$,

$$
\begin{aligned}
\tilde{F} \wedge \tilde{F} = {} & (E^1\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x + E^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y + E^3\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}z + B^3\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y - B^2\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z + B^1\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z) \\
& \wedge (E^1\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x + E^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y + E^3\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}z + B^3\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y - B^2\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z + B^1\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z) \\
= {} & E^1B^1\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z - E^2B^2\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z + E^3B^3\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}z \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \\
& + B^3E^3\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}z - B^2E^2\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z \wedge \tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}y + B^1E^1\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z \wedge \tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \\
= {} & (E^1B^1 + E^2B^2 + E^3B^3 + B^1E^1 + B^2E^2 + B^3E^3)\,\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z \\
= {} & (2\boldsymbol{B} \cdot \boldsymbol{E})\,\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z
\end{aligned}
$$

Therefore, $\displaystyle\int \tilde{F} \wedge \tilde{F} = \int (2\boldsymbol{B} \cdot \boldsymbol{E})\,\tilde{\mathrm{d}}t \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z = \int (2\boldsymbol{B} \cdot \boldsymbol{E})\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$

Question 3

3a. We recall that for a scalar field f, $\pounds_{\bar{V}} f = \bar{V} f$, and for a vector field $\bar{W}$, $\pounds_{\bar{V}} \bar{W} = \left[\bar{V}, \bar{W}\right]$. By the result of question 1(a), we have $\left[\bar{V}, \bar{W}\right]^i = V^j \frac{\partial}{\partial x^j} W^i - W^j \frac{\partial}{\partial x^j} V^i$.

Consider an arbitrary vector field $\bar{W}$. By product rule, $\pounds_{\bar{V}} \left(\tilde{U}(\bar{W})\right) = (\pounds_{\bar{V}} \tilde{U})(\bar{W}) + \tilde{U}(\pounds_{\bar{V}} \bar{W})$, and therefore

$$\begin{aligned}
(\pounds_{\bar{V}} \tilde{U})(\bar{W}) &= \pounds_{\bar{V}} \left(\tilde{U}(\bar{W})\right) - \tilde{U}(\pounds_{\bar{V}} \bar{W}) \\
&= \bar{V} \left(\tilde{U}(\bar{W})\right) - \tilde{U}(\left[\bar{V}, \bar{W}\right]) \\
&= V^j \frac{\partial}{\partial x^j} \left(U_i W^i\right) - U_i \left[\bar{V}, \bar{W}\right]^i \\
&= V^j \left(\cancel{U_i \frac{\partial}{\partial x^j} W^i} + W^i \frac{\partial}{\partial x^j} U_i\right) - U_i \left(\cancel{V^j \frac{\partial}{\partial x^j} W^i} - W^j \frac{\partial}{\partial x^j} V^i\right) \\
&= V^j W^i \frac{\partial}{\partial x^j} U_i + U_i W^j \frac{\partial}{\partial x^j} V^i \\
&= \left(V^j \frac{\partial}{\partial x^j} U_i + U_j \frac{\partial}{\partial x^i} V^j\right) W^i, \text{ relabelling summation indices on second term}
\end{aligned}$$

Since the LHS of the expression can also be written as $(\pounds_{\bar{V}} \tilde{U})(\bar{W}) = (\pounds_{\bar{V}} \tilde{U})_i W^i$ and $\bar{W}$ was arbitrary, we hence conclude that $(\pounds_{\bar{V}} \tilde{U})_i = V^j \frac{\partial}{\partial x^j} U_i + U_j \frac{\partial}{\partial x^i} V^j$.

(Alternative solution) We can directly use a basis vector $\bar{e}_i$ instead of $\bar{W}$, obtaining

$$\begin{aligned}
(\pounds_{\bar{V}} \tilde{U})_i = (\pounds_{\bar{V}} \tilde{U})(\bar{e}_i) &= \pounds_{\bar{V}} \left(\tilde{U}(\bar{e}_i)\right) - \tilde{U}(\pounds_{\bar{V}} \bar{e}_i) \\
&= \bar{V}\left(U_i\right) - U_j \left[\bar{V}, \bar{e}_i\right]^j \\
&= V^j \frac{\partial}{\partial x^j} U_i - U_j \left(V^k \frac{\partial}{\partial x^k} (\bar{e}_i)^j - (\bar{e}_i)^k \frac{\partial}{\partial x^k} V^j\right) \\
&= V^j \frac{\partial}{\partial x^j} U_i - U_j \left(V^k \frac{\partial}{\partial x^k} \delta_i^j - \delta_i^k \frac{\partial}{\partial x^k} V^j\right) \\
&= V^j \frac{\partial}{\partial x^j} U_i - U_j \left(0 - \frac{\partial}{\partial x^i} V^j\right) \\
&= V^j \frac{\partial}{\partial x^j} U_i + U_j \frac{\partial}{\partial x^i} V^j
\end{aligned}$$

As for $(\pounds_{\bar{V}}\tilde{\omega})_{ij}$, similarly by product rule we have $\pounds_{\bar{V}}\left(\tilde{\omega}(\bar{W},\bar{X})\right) = (\pounds_{\bar{V}}\tilde{\omega})(\bar{W},\bar{X}) + \tilde{\omega}(\pounds_{\bar{V}}\bar{W},\bar{X}) + \tilde{\omega}(\bar{W},\pounds_{\bar{V}}\bar{X})$ for arbitrary vector fields $\bar{W}$, $\bar{X}$. Therefore,

$$
\begin{aligned}
(\pounds_{\bar{V}}\tilde{\omega})(\bar{W},\bar{X}) &= \pounds_{\bar{V}}\left(\tilde{\omega}(\bar{W},\bar{X})\right) - \tilde{\omega}(\pounds_{\bar{V}}\bar{W},\bar{X}) - \tilde{\omega}(\bar{W},\pounds_{\bar{V}}\bar{X}) \\
&= V^k\frac{\partial}{\partial x^k}\left(\omega_{ij}W^iX^j\right) - \omega_{ij}\left[\bar{V},\bar{W}\right]^i X^j - \omega_{ij}W^i\left[\bar{V},\bar{X}\right]^j \\
&= V^k\left(\frac{\partial\omega_{ij}}{\partial x^k}W^iX^j + \cancel{\omega_{ij}\frac{\partial W^i}{\partial x^k}X^j} + \cancel{\omega_{ij}W^i\frac{\partial X^j}{\partial x^k}}\right) - \omega_{ij}\left(\cancel{V^k\frac{\partial W^i}{\partial x^k}} - W^k\frac{\partial V^i}{\partial x^k}\right)X^j \\
&\quad - \omega_{ij}W^i\left(\cancel{V^k\frac{\partial X^j}{\partial x^k}} - X^k\frac{\partial V^j}{\partial x^k}\right) \\
&= V^k\frac{\partial\omega_{ij}}{\partial x^k}W^iX^j + \omega_{ij}W^k\frac{\partial V^i}{\partial x^k}X^j + \omega_{ij}W^iX^k\frac{\partial V^j}{\partial x^k} \\
&= V^k\frac{\partial\omega_{ij}}{\partial x^k}W^iX^j + \omega_{kj}W^i\frac{\partial V^k}{\partial x^i}X^j + \omega_{ik}W^iX^j\frac{\partial V^k}{\partial x^j}, \text{ relabelling summation indices} \\
&= \left(V^k\frac{\partial}{\partial x^k}\omega_{ij} + \omega_{kj}\frac{\partial}{\partial x^i}V^k + \omega_{ik}\frac{\partial}{\partial x^j}V^k\right)W^iX^j
\end{aligned}
$$

Since the LHS of the expression can also be written as $(\pounds_{\bar{V}}\tilde{\omega})(\bar{W},\bar{X}) = (\pounds_{\bar{V}}\tilde{\omega})_{ij}W^iX^j$ and $\bar{W},\bar{X}$ were arbitrary, we hence conclude that $(\pounds_{\bar{V}}\tilde{\omega})_{ij} = V^k\frac{\partial}{\partial x^k}\omega_{ij} + \omega_{kj}\frac{\partial}{\partial x^i}V^k + \omega_{ik}\frac{\partial}{\partial x^j}V^k$.

(As with the first part, an alternative solution is also possible by using $(\pounds_{\bar{V}}\tilde{\omega})_{ij} = (\pounds_{\bar{V}}\tilde{\omega})(\bar{e}_i,\bar{e}_j)$ directly instead of $\bar{W},\bar{X}$.)

3bi. Note that since we are working in Euclidean space, the time t is a parameter rather than a coordinate in this context. The Euclidean metric in standard coordinates is $g_{ij} = \delta_{ij}$.

The given expression for the Euler equation is in the form of vector components, while the desired result is a 1-form equation. We hence begin by lowering the free index i:

$$
\begin{aligned}
&g_{ik}\frac{\partial v^i}{\partial t} + g_{ik}v^j\frac{\partial v^i}{\partial x^j} = -g_{ik}\delta^{ij}\frac{\partial p}{\partial x^j} \\
&\frac{\partial(g_{ik}v^i)}{\partial t} + v^j\frac{\partial(g_{ik}v^i)}{\partial x^j} = -\frac{\partial p}{\partial x^k}, \text{ since } g_{ij} = \delta_{ij} \text{ is a constant w.r.t. } t \text{ and the coordinates } x^j \\
&\frac{\partial v_k}{\partial t} + v^j\frac{\partial v_k}{\partial x^j} = -\frac{\partial p}{\partial x^k}
\end{aligned}
$$

(Some care needs to be taken with the step $-g_{ik}\delta^{ij}\frac{\partial p}{\partial x^j} = -\frac{\partial p}{\partial x^k}$, because the Kronecker delta δ^{ij} with two upper indices is not tensorial. However, viewing $-\delta^{ij}\frac{\partial p}{\partial x^j}$ as giving the components of a vector whose i^{th} component is $-\frac{\partial p}{\partial x^i}$ in standard coordinates, it can be seen that the corresponding one-form under the Euclidean metric in standard coordinates (which is simply the identity matrix) has i^{th} component $-\frac{\partial p}{\partial x^i}$ as well.)

Taking a 1-form coordinate basis $\{\tilde{\omega}^k\}$ and summing over the components in the above equation, we have

$$\left(\frac{\partial v_k}{\partial t} + v^j \frac{\partial v_k}{\partial x^j}\right) \tilde{\omega}^k = \left(-\frac{\partial p}{\partial x^k}\right) \tilde{\omega}^k$$

$$\frac{\partial v_k}{\partial t}\tilde{\omega}^k + \left(v^j \frac{\partial v_k}{\partial x^j} + v^j \frac{\partial v_j}{\partial x^k}\right) \tilde{\omega}^k = v^j \frac{\partial v_j}{\partial x^k}\tilde{\omega}^k - \frac{\partial p}{\partial x^k}\tilde{\omega}^k, \text{ adding } v^j \frac{\partial v_j}{\partial x^k}\tilde{\omega}^k \text{ to both sides}$$

$$\frac{\partial v_k}{\partial t}\tilde{\omega}^k + (\pounds_{\bar{v}}\tilde{v})_k\tilde{\omega}^k = v^j \frac{\partial v_j}{\partial x^k}\tilde{\omega}^k - \tilde{\mathrm{d}}p, \text{ using the result in part (a) and noting } \tilde{\mathrm{d}}p = \frac{\partial p}{\partial x^k}\tilde{\omega}^k$$

$$\frac{\partial (v_k\tilde{\omega}^k)}{\partial t} + \pounds_{\bar{v}}\tilde{v} = \frac{1}{2}\left(v^j \frac{\partial v_j}{\partial x^k} + v^j \frac{\partial v_j}{\partial x^k}\right) \tilde{\omega}^k - \tilde{\mathrm{d}}p, \text{ since the basis 1-forms } \tilde{\omega}^k \text{ are independent of } t$$

$$\frac{\partial \tilde{v}}{\partial t} + \pounds_{\bar{v}}\tilde{v} = \frac{1}{2}\left(v^j \frac{\partial v_j}{\partial x^k} + v^j \frac{\partial (g_{ij}v^i)}{\partial x^k}\right) \tilde{\omega}^k - \tilde{\mathrm{d}}p$$

$$\frac{\partial \tilde{v}}{\partial t} + \pounds_{\bar{v}}\tilde{v} = \frac{1}{2}\left(v^j \frac{\partial v_j}{\partial x^k} + v^j g_{ij} \frac{\partial v^i}{\partial x^k}\right) \tilde{\omega}^k - \tilde{\mathrm{d}}p, \text{ since } g_{ij} = \delta_{ij} \text{ is constant w.r.t. the coordinates}$$

$$\frac{\partial \tilde{v}}{\partial t} + \pounds_{\bar{v}}\tilde{v} = \frac{1}{2}\left(v^j \frac{\partial v_j}{\partial x^k} + v_i \frac{\partial v^i}{\partial x^k}\right) \tilde{\omega}^k - \tilde{\mathrm{d}}p$$

$$\frac{\partial \tilde{v}}{\partial t} + \pounds_{\bar{v}}\tilde{v} = \frac{1}{2}\left(\frac{\partial (v^j v_j)}{\partial x^k}\right) \tilde{\omega}^k - \tilde{\mathrm{d}}p, \text{ by product rule}$$

$$\frac{\partial \tilde{v}}{\partial t} + \pounds_{\bar{v}}\tilde{v} = \frac{1}{2}\tilde{\mathrm{d}}\left(v^j v_j\right) - \tilde{\mathrm{d}}p$$

$$\left(\frac{\partial}{\partial t} + \pounds_{\bar{v}}\right)\tilde{v} = \tilde{\mathrm{d}}\left(\frac{1}{2}|\bar{v}|^2 - p\right), \text{ since } |\bar{v}|^2 = g_{ij}v^i v^j = v^j v_j \text{ by definition}$$

(Practically speaking, it may be easier to begin from the desired expression, writing $\tilde{v}$ as $v_k\tilde{\omega}^k$, then working backwards to the given expression.)

3bii. We first note that similarly to the previous part, we can use the fact that the Euclidean metric $g_{ij} = \delta_{ij}$ is independent of the coordinates to write

$$\frac{\partial v_k}{\partial x^i}\frac{\partial v^k}{\partial x^j} = \frac{\partial v_k}{\partial x^i}\frac{\partial (g^{kl}v_l)}{\partial x^j} = \frac{\partial v_k}{\partial x^i}g^{kl}\frac{\partial v_l}{\partial x^j} = \frac{\partial (g^{kl}v_k)}{\partial x^i}\frac{\partial v_l}{\partial x^j} = \frac{\partial v^l}{\partial x^i}\frac{\partial v_l}{\partial x^j},$$

i.e. we can "swap" the raised and lowered indices in the two terms. With this, we now show that $\pounds_{\bar{v}}(\tilde{\mathrm{d}}\tilde{v}) = \tilde{\mathrm{d}}\left(\pounds_{\bar{v}}\tilde{v}\right)$:

$$\left(\pounds_{\bar{v}}(\tilde{\mathrm{d}}\tilde{v})\right)_{ij} = v^k\frac{\partial}{\partial x^k}(\tilde{\mathrm{d}}\tilde{v})_{ij} + (\tilde{\mathrm{d}}\tilde{v})_{kj}\frac{\partial}{\partial x^i}v^k + (\tilde{\mathrm{d}}\tilde{v})_{ik}\frac{\partial}{\partial x^j}v^k, \text{ using the result of part (a)}$$

$$= v^k\frac{\partial}{\partial x^k}((1+1)\partial_{[i}v_{j]}) + ((1+1)\partial_{[k}v_{j]})\frac{\partial v^k}{\partial x^i} + ((1+1)\partial_{[i}v_{k]})\frac{\partial v^k}{\partial x^j}$$

$$= 2\left(v^k\frac{\partial}{\partial x^k}\left(\frac{1}{2!}\left(\frac{\partial v_j}{\partial x^i} - \frac{\partial v_i}{\partial x^j}\right)\right) + \left(\frac{1}{2!}\left(\frac{\partial v_j}{\partial x^k} - \cancel{\frac{\partial v_k}{\partial x^j}}\right)\right)\frac{\partial v^k}{\partial x^i} + \left(\frac{1}{2!}\left(\cancel{\frac{\partial v_k}{\partial x^i}} - \frac{\partial v_i}{\partial x^k}\right)\right)\frac{\partial v^k}{\partial x^j}\right)$$

(the cancellation makes use of the $\frac{\partial v_k}{\partial x^i}\frac{\partial v^k}{\partial x^j} = \frac{\partial v^k}{\partial x^i}\frac{\partial v_k}{\partial x^j}$ equality derived above)

$$= v^k\frac{\partial}{\partial x^k}\left(\frac{\partial v_j}{\partial x^i} - \frac{\partial v_i}{\partial x^j}\right) + \frac{\partial v_j}{\partial x^k}\frac{\partial v^k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\frac{\partial v^k}{\partial x^j}$$

$$
\begin{aligned}
\left(\tilde{\mathrm{d}}\left(\pounds_{\bar{v}}\tilde{v}\right)\right)_{ij} &= (1+1)\partial_{[i}(\pounds_{\bar{v}}\tilde{v})_{j]} \\
&= 2\left(\frac{1}{2!}\left(\frac{\partial}{\partial x^i}(\pounds_{\bar{v}}\tilde{v})_j - \frac{\partial}{\partial x^j}(\pounds_{\bar{v}}\tilde{v})_i\right)\right) \\
&= \frac{\partial}{\partial x^i}\left(v^k\frac{\partial v_j}{\partial x^k} + v_k\frac{\partial v^k}{\partial x^j}\right) - \frac{\partial}{\partial x^j}\left(v^k\frac{\partial v_i}{\partial x^k} + v_k\frac{\partial v^k}{\partial x^i}\right)\text{, using the result of part (a)} \\
&= \frac{\partial v^k}{\partial x^i}\frac{\partial v_j}{\partial x^k} + v^k\frac{\partial}{\partial x^i}\frac{\partial v_j}{\partial x^k} + \cancel{\frac{\partial v_k}{\partial x^i}\frac{\partial v^k}{\partial x^j}} + v_k\frac{\partial}{\partial x^i}\frac{\partial v^k}{\partial x^j} - \left(\frac{\partial v^k}{\partial x^j}\frac{\partial v_i}{\partial x^k} + v^k\frac{\partial}{\partial x^j}\frac{\partial v_i}{\partial x^k}\right. \\
&\qquad \left. + \cancel{\frac{\partial v_k}{\partial x^j}\frac{\partial v^k}{\partial x^i}} + v_k\frac{\partial}{\partial x^j}\frac{\partial v^k}{\partial x^i}\right)\text{, again using } \frac{\partial v_k}{\partial x^i}\frac{\partial v^k}{\partial x^j} = \frac{\partial v^k}{\partial x^i}\frac{\partial v_k}{\partial x^j}\text{ to cancel terms} \\
&= v^k\frac{\partial}{\partial x^k}\frac{\partial v_j}{\partial x^i} - v^k\frac{\partial}{\partial x^k}\frac{\partial v_i}{\partial x^j} + \frac{\partial v^k}{\partial x^i}\frac{\partial v_j}{\partial x^k} - \frac{\partial v^k}{\partial x^j}\frac{\partial v_i}{\partial x^k} + \cancel{v_k\frac{\partial}{\partial x^i}\frac{\partial v^k}{\partial x^j}} - \cancel{v_k\frac{\partial}{\partial x^j}\frac{\partial v^k}{\partial x^i}}, \\
&\quad \text{assuming partial derivatives commute} \\
&= v^k\frac{\partial}{\partial x^k}\left(\frac{\partial v_j}{\partial x^i} - \frac{\partial v_i}{\partial x^j}\right) + \frac{\partial v^k}{\partial x^i}\frac{\partial v_j}{\partial x^k} - \frac{\partial v^k}{\partial x^j}\frac{\partial v_i}{\partial x^k} \\
&= \left(\pounds_{\bar{v}}(\tilde{\mathrm{d}}\tilde{v})\right)_{ij}\text{, as shown above}
\end{aligned}
$$

Therefore, $\pounds_{\bar{v}}(\tilde{\mathrm{d}}\tilde{v}) = \tilde{\mathrm{d}}\left(\pounds_{\bar{v}}\tilde{v}\right)$. Also, we note that $\frac{\partial}{\partial t}$ and $\tilde{\mathrm{d}}$ should commute (i.e. $\frac{\partial}{\partial t}(\tilde{\mathrm{d}}\tilde{v}) = \tilde{\mathrm{d}}\left(\frac{\partial}{\partial t}\tilde{v}\right)$) because t is independent of the coordinates while the exterior derivative $\tilde{\mathrm{d}}$ only takes derivatives with respect to the coordinates. Hence, we can conclude that

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \pounds_{\bar{v}}\right)\tilde{\mathrm{d}}\tilde{v} = \frac{\partial}{\partial t}(\tilde{\mathrm{d}}\tilde{v}) + \pounds_{\bar{v}}(\tilde{\mathrm{d}}\tilde{v}) &= \tilde{\mathrm{d}}\left(\frac{\partial}{\partial t}\tilde{v} + \pounds_{\bar{v}}\tilde{v}\right) \quad \text{since } \tilde{\mathrm{d}} \text{ commutes with } \frac{\partial}{\partial t} \text{ and } \pounds_{\bar{v}} \\
&= \tilde{\mathrm{d}}\left(\tilde{\mathrm{d}}\left(\frac{1}{2}|\bar{v}|^2 - p\right)\right) \quad \text{by the Euler equation} \\
&= 0 \quad \text{since } \tilde{\mathrm{d}}^2 = 0
\end{aligned}
$$

Solutions provided by: Ernest Tan