NATIONAL UNIVERSITY OF SINGAPORE

PC4274 MATHEMATICAL METHODS IN PHYSICS III

(Semester II: AY 2013–14)

Time Allowed: 2 Hours

INSTRUCTIONS TO STUDENTS

1. Write your matriculation number only. Do not write your name.
2. This examination paper contains **THREE** questions and comprises **THREE** printed pages.
3. Students are required to answer **ALL** questions.
4. Students should write the answers for each question on a new page.
5. This is a CLOSED BOOK examination.
6. The total mark for this examination paper is 100.

1. Let M be an n-dimensional manifold with coordinates x^i, $i = 1, 2, \ldots, n$.

(a) If $\bar{X}$ and $\bar{Y}$ are vector fields on M, show that $[\bar{X}, \bar{Y}] \equiv \bar{X}\bar{Y} - \bar{Y}\bar{X}$ is a vector field with components

$$[\bar{X}, \bar{Y}]^i = X^j \frac{\partial}{\partial x^j} Y^i - Y^j \frac{\partial}{\partial x^j} X^i$$

on a coordinate basis.

(b) If $\tilde{\alpha}$ is a one-form field on M, show that $\tilde{d}\tilde{\alpha}$ is a two-form field with components

$$(\tilde{d}\tilde{\alpha})_{ij} = \frac{\partial \alpha_j}{\partial x_i} - \frac{\partial \alpha_i}{\partial x_j}$$

on a coordinate basis.

(c) Hence show that

$$\tilde{d}\tilde{\alpha}(\bar{X}, \bar{Y}) = \bar{X}\tilde{\alpha}(\bar{Y}) - \bar{Y}\tilde{\alpha}(\bar{X}) - \tilde{\alpha}([\bar{X}, \bar{Y}])\,.$$

If $\tilde{\alpha}$ is exact, verify that the right-hand side of this equation is zero.

[30 marks]

2. Consider four-dimensional Minkowski space with coordinates $x^\mu = (t, x, y, z)$.

(a) Write down an explicit coordinate basis for two-forms at a point in Minkowski space. What is the dimension of this vector space of two-forms? What does this space map to under the dual map $*$?

(b) The electromagnetic field strength $\tilde{F}$ is a two-form field in Minkowski space with components

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix}$$

on a coordinate basis.

(i) Calculate $*\tilde{F}$ and find its components on a coordinate basis. What action does the dual map have on the electric and magnetic fields?

(ii) Calculate $\tilde{F} \wedge *\tilde{F}$ and $\tilde{F} \wedge \tilde{F}$. Hence show that

$$\int \tilde{F} \wedge *\tilde{F} = \int (|\boldsymbol{B}|^2 - |\boldsymbol{E}|^2)\ \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\,,$$

and find the corresponding expression for $\int \tilde{F} \wedge \tilde{F}$.

[*Hint: Recall* $*(\tilde{\mathrm{d}}x^{\mu_1} \wedge \cdots \wedge \tilde{\mathrm{d}}x^{\mu_p}) = \frac{1}{(4-p)!}\, \epsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_4} \tilde{\mathrm{d}}x^{\mu_{p+1}} \wedge \cdots \wedge \tilde{\mathrm{d}}x^{\mu_4}$.]

[30 marks]

3. (a) Let M be an n-dimensional manifold with coordinates x^i. If $\tilde{U}$ is a one-form field and $\tilde{\omega}$ a two-form field on M, show that their Lie derivatives along a vector field $\bar{V}$ have components

$$(\pounds_{\bar{V}}\tilde{U})_i = V^j \frac{\partial}{\partial x^j}\, U_i + U_j \frac{\partial}{\partial x^i}\, V^j,$$

$$(\pounds_{\bar{V}}\tilde{\omega})_{ij} = V^k \frac{\partial}{\partial x^k}\, \omega_{ij} + \omega_{kj} \frac{\partial}{\partial x^i}\, V^k + \omega_{ik} \frac{\partial}{\partial x^j}\, V^k,$$

respectively, on a coordinate basis. [*Hint: You may need the components of* $\pounds_{\bar{V}}\bar{W} = [\bar{V}, \bar{W}]$ *found in Question 1(a).*]

(b) The Euler equation for an incompressible fluid in Euclidean space is

$$\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} = -\delta^{ij} \frac{\partial p}{\partial x^j}\,,$$

where v^i is the velocity field of the fluid, and p its pressure field (a scalar field).

(i) Show that the Euler equation can be written as the one-form equation:

$$\left(\frac{\partial}{\partial t} + \pounds_{\bar{v}}\right)\tilde{v} = \tilde{\mathrm{d}}\left(\tfrac{1}{2}|\bar{v}|^2 - p\right).$$

(ii) Hence deduce that

$$\left(\frac{\partial}{\partial t} + \pounds_{\bar{v}}\right)\tilde{\mathrm{d}}\tilde{v} = 0\,,$$

where $\tilde{\mathrm{d}}\tilde{v}$ is known as the vorticity two-form. [*Hint: Show that* $\pounds_{\bar{v}}(\tilde{\mathrm{d}}\tilde{v}) = \tilde{\mathrm{d}}(\pounds_{\bar{v}}\tilde{v})$.]

[40 marks]

(ET)

– END OF PAPER –