PC4274 Mathematical Methods in Physics III

Exam Answers

1. (a) (i)

$$\begin{aligned}\tilde{\omega}(\bar{V}) &= (-t\,\tilde{\mathrm{d}}t + x\,\tilde{\mathrm{d}}x)\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right)\\ &= -t\,\tilde{\mathrm{d}}t\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right) + x\,\tilde{\mathrm{d}}x\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right)\\ &= -t\,\tilde{\mathrm{d}}t\left(t\frac{\partial}{\partial t}\right) + x\,\tilde{\mathrm{d}}x\left(x\frac{\partial}{\partial x}\right)\\ &= -t^2 + x^2\end{aligned}$$

(ii)

$$\begin{aligned}\boldsymbol{g}(\bar{V},\,\cdot\,) &= (-\,\tilde{\mathrm{d}}t\otimes\tilde{\mathrm{d}}t + \tilde{\mathrm{d}}x\otimes\tilde{\mathrm{d}}x + \tilde{\mathrm{d}}y\otimes\tilde{\mathrm{d}}y + \tilde{\mathrm{d}}z\otimes\tilde{\mathrm{d}}z)(\bar{V},\,\cdot\,)\\ &= -\,\tilde{\mathrm{d}}t(\bar{V})\,\tilde{\mathrm{d}}t + \tilde{\mathrm{d}}x(\bar{V})\,\tilde{\mathrm{d}}x + \tilde{\mathrm{d}}y(\bar{V})\,\tilde{\mathrm{d}}y + \tilde{\mathrm{d}}z(\bar{V})\,\tilde{\mathrm{d}}z\\ &= -t\,\tilde{\mathrm{d}}t + x\,\tilde{\mathrm{d}}x\end{aligned}$$

This is the same as $\tilde{\omega}$.

(iii)

$$\begin{aligned}\boldsymbol{g}(\bar{V},\bar{V}) &= -t\,\tilde{\mathrm{d}}t(\bar{V}) + x\,\tilde{\mathrm{d}}x(\bar{V})\\ &= -t^2 + x^2\end{aligned}$$

This is the same as $\tilde{\omega}(\bar{V})$.

(b) Now we have

$$\tilde{\mathrm{d}}f = \frac{\partial f}{\partial t}\,\tilde{\mathrm{d}}t + \frac{\partial f}{\partial x}\,\tilde{\mathrm{d}}x$$

Thus

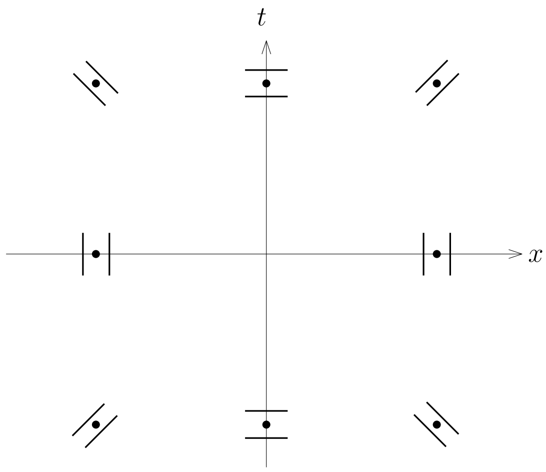
$$\frac{\partial f}{\partial t} = -t \qquad \text{and} \qquad \frac{\partial f}{\partial x} = x$$

These two equations can be straightforwardly integrated to obtain

$$f = \tfrac{1}{2}(-t^2 + x^2),$$

up to an additive constant.

Sketch of $\tilde{\omega}$:



2. (a) Under a general coordinate transformation $y^{i'} = y^{i'}(x^i)$, we have

$$B_{i'j'} = \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} B_{ij}$$

Note that

$$\begin{aligned} B_{j'i'} &= \frac{\partial x^i}{\partial y^{j'}} \frac{\partial x^j}{\partial y^{i'}} B_{ij} \\ &= \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^i}{\partial y^{i'}} B_{ji} \\ &= -\frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} B_{ij} \\ &= -B_{i'j'} \end{aligned}$$

Thus $B_{i'j'}$ is also antisymmetric.

(b) Under the same coordinate transformation, we have

$$\begin{aligned} \frac{\partial B_{j'k'}}{\partial y^{i'}} &= \frac{\partial}{\partial y^{i'}} \left(\frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} B_{jk} \right) \\ &= \frac{\partial x^i}{\partial y^{i'}} \frac{\partial}{\partial x^i} \left(\frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} B_{jk} \right) \\ &= \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} \frac{\partial B_{jk}}{\partial x^i} + \frac{\partial^2 x^j}{\partial y^{i'} \partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} B_{jk} + \frac{\partial x^j}{\partial y^{j'}} \frac{\partial^2 x^k}{\partial y^{i'} \partial y^{k'}} B_{jk} \end{aligned}$$

The first term has the correct form for the transformation law of a $\binom{0}{3}$ tensor. However, the presence of the second and third terms means that $\partial_i B_{jk}$ is not a $\binom{0}{3}$ tensor.

(c)

$$
\begin{aligned}
H_{i'j'k'} &= \partial_{i'} B_{j'k'} + \partial_{j'} B_{k'i'} + \partial_{k'} B_{i'j'} \\
&= \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} \frac{\partial B_{jk}}{\partial x^i} + \frac{\partial^2 x^j}{\partial y^{i'} \partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} B_{jk} + \frac{\partial x^j}{\partial y^{j'}} \frac{\partial^2 x^k}{\partial y^{i'} \partial y^{k'}} B_{jk} \\
&\quad + \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} \frac{\partial x^i}{\partial y^{i'}} \frac{\partial B_{ki}}{\partial x^j} + \frac{\partial^2 x^k}{\partial y^{j'} \partial y^{k'}} \frac{\partial x^i}{\partial y^{i'}} B_{ki} + \frac{\partial x^k}{\partial y^{k'}} \frac{\partial^2 x^i}{\partial y^{j'} \partial y^{i'}} B_{ki} \\
&\quad + \frac{\partial x^k}{\partial y^{k'}} \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial B_{ij}}{\partial x^k} + \frac{\partial^2 x^i}{\partial y^{k'} \partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} B_{ij} + \frac{\partial x^i}{\partial y^{i'}} \frac{\partial^2 x^j}{\partial y^{k'} \partial y^{j'}} B_{ij} \\
&= \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} \left(\frac{\partial B_{jk}}{\partial x^i} + \frac{\partial B_{ki}}{\partial x^j} + \frac{\partial B_{ij}}{\partial x^k} \right) \\
&\quad + \frac{\partial x^k}{\partial y^{k'}} \left[\frac{\partial^2 x^j}{\partial y^{i'} \partial y^{j'}} B_{jk} + \frac{\partial^2 x^i}{\partial y^{j'} \partial y^{i'}} B_{ki} \right] \\
&\quad + \frac{\partial x^j}{\partial y^{j'}} \left[\frac{\partial^2 x^k}{\partial y^{i'} \partial y^{k'}} B_{jk} + \frac{\partial^2 x^i}{\partial y^{k'} \partial y^{i'}} B_{ij} \right] \\
&\quad + \frac{\partial x^i}{\partial y^{i'}} \left[\frac{\partial^2 x^k}{\partial y^{j'} \partial y^{k'}} B_{ki} + \frac{\partial^2 x^j}{\partial y^{k'} \partial y^{j'}} B_{ij} \right] \\
&= \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} H_{ijk}
\end{aligned}
$$

In the second last step, the terms in each of the square brackets cancel after renaming of dummy indices, using the anti-symmetry of B_{ij} and the commutativity of partial derivatives. So H_{ijk} transforms as a $\binom{0}{3}$ tensor.

3. (a)

$$
\begin{aligned}
\tilde{\mathrm{d}}\tilde{\alpha} &= \left(\frac{\partial P}{\partial x} \tilde{\mathrm{d}}x + \frac{\partial P}{\partial y} \tilde{\mathrm{d}}y + \frac{\partial P}{\partial z} \tilde{\mathrm{d}}z \right) \wedge \tilde{\mathrm{d}}x + \left(\frac{\partial Q}{\partial x} \tilde{\mathrm{d}}x + \frac{\partial Q}{\partial y} \tilde{\mathrm{d}}y + \frac{\partial Q}{\partial z} \tilde{\mathrm{d}}z \right) \wedge \tilde{\mathrm{d}}y \\
&\quad + \left(\frac{\partial R}{\partial x} \tilde{\mathrm{d}}x + \frac{\partial R}{\partial y} \tilde{\mathrm{d}}y + \frac{\partial R}{\partial z} \tilde{\mathrm{d}}z \right) \wedge \tilde{\mathrm{d}}z \\
&= \frac{\partial P}{\partial y} \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}x + \frac{\partial P}{\partial z} \tilde{\mathrm{d}}z \wedge \tilde{\mathrm{d}}x + \frac{\partial Q}{\partial x} \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y + \frac{\partial Q}{\partial z} \tilde{\mathrm{d}}z \wedge \tilde{\mathrm{d}}y \\
&\quad + \frac{\partial R}{\partial x} \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}z + \frac{\partial R}{\partial y} \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \tilde{\mathrm{d}}z \wedge \tilde{\mathrm{d}}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathrm{d}}^2\tilde{\alpha} &= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z \wedge \tilde{\mathrm{d}}x \\
&\quad + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\tilde{\mathrm{d}}z \wedge \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \\
&= \left(\frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y}\right)\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z \\
&= 0
\end{aligned}
$$

(b) Stokes' theorem:

$$\int_C \tilde{\alpha} = \int_A \tilde{\mathrm{d}}\tilde{\alpha}$$

where C is the boundary of the surface A. Using Eq. (1),

$$
\begin{aligned}
&\int_C P\,\tilde{\mathrm{d}}x + Q\,\tilde{\mathrm{d}}y + R\,\tilde{\mathrm{d}}z = \\
&\qquad \int_A \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\tilde{\mathrm{d}}z \wedge \tilde{\mathrm{d}}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y
\end{aligned}
$$

For $P = z(x^2 - 1)$, $Q = 0$, $R = y(x+1)$, we obtain

$$\int_C z(x^2-1)\,\tilde{\mathrm{d}}x + y(x+1)\,\tilde{\mathrm{d}}z = \int_A (x+1)\,\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z + (x^2 - y - 1)\,\tilde{\mathrm{d}}z \wedge \tilde{\mathrm{d}}x$$

We can choose A to be the interior of the unit circle C. Since the surface A lies in the $z = 0$ plane, the 'tubes' of $\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z$ and $\tilde{\mathrm{d}}z \wedge \tilde{\mathrm{d}}x$ do not cut through A. Hence the surface integral on the RHS is zero.

4. (a) By the Leibniz rule

$$
\begin{aligned}
(\pounds_{\bar{V}}\tilde{\omega})(\bar{W}) &= \pounds_{\bar{V}}(\tilde{\omega}(\bar{W})) - \tilde{\omega}(\pounds_{\bar{V}}\bar{W}) \\
&= \bar{V}(\tilde{\omega}(\bar{W})) - \tilde{\omega}(\pounds_{\bar{V}}\bar{W})
\end{aligned}
$$

In a coordinate basis, we have

$$
\begin{aligned}
(\pounds_{\bar{V}}\tilde{\omega})_i W^i &= V^j \frac{\partial}{\partial x^j}(\omega_i W^i) - \omega_i (\pounds_{\bar{V}}\bar{W})^i \\
&= V^j \frac{\partial \omega_i}{\partial x^j} W^i + \cancel{V^j \omega_i \frac{\partial W^i}{\partial x^j}} - \omega_i \left(\cancel{V^j \frac{\partial W^i}{\partial x^j}} - W^j \frac{\partial V^i}{\partial x^j}\right) \\
&= V^j \frac{\partial \omega_i}{\partial x^j} W^i + \omega_j \frac{\partial V^j}{\partial x^i} W^i
\end{aligned}
$$

Since this is true for any W^i,

$$(\pounds_{\bar{V}}\tilde{\omega})_i = V^j \frac{\partial \omega_i}{\partial x^j} + \omega_j \frac{\partial V^j}{\partial x^i}$$

(b) Since $\tilde{\omega} \wedge \tilde{\sigma} = \tilde{\omega} \otimes \tilde{\sigma} - \tilde{\sigma} \otimes \tilde{\omega}$, we have

$$\begin{aligned}
\pounds_{\bar{V}}(\tilde{\omega} \wedge \tilde{\sigma}) &= \pounds_{\bar{V}}(\tilde{\omega} \otimes \tilde{\sigma} - \tilde{\sigma} \otimes \tilde{\omega}) \\
&= \pounds_{\bar{V}}(\tilde{\omega} \otimes \tilde{\sigma}) - \pounds_{\bar{V}}(\tilde{\sigma} \otimes \tilde{\omega})
\end{aligned}$$

To evaluate the individual terms on the RHS, we use the Leibniz rule:

$$\begin{aligned}
\big(\pounds_{\bar{V}}(\tilde{\omega} \otimes \tilde{\sigma})\big)(\bar{X}, \bar{Y}) &= \pounds_{\bar{V}}\big((\tilde{\omega} \otimes \tilde{\sigma})(\bar{X}, \bar{Y})\big) - (\tilde{\omega} \otimes \tilde{\sigma})(\pounds_{\bar{V}}\bar{X}, \bar{Y}) - (\tilde{\omega} \otimes \tilde{\sigma})(\bar{X}, \pounds_{\bar{V}}\bar{Y}) \\
&= \pounds_{\bar{V}}\big(\tilde{\omega}(\bar{X})\tilde{\sigma}(\bar{Y})\big) - \tilde{\omega}(\pounds_{\bar{V}}\bar{X})\tilde{\sigma}(\bar{Y}) - \tilde{\omega}(\bar{X})\tilde{\sigma}(\pounds_{\bar{V}}\bar{Y}) \\
&= \big[\pounds_{\bar{V}}\big(\tilde{\omega}(\bar{X})\big) - \tilde{\omega}(\pounds_{\bar{V}}\bar{X})\big]\tilde{\sigma}(\bar{Y}) + \tilde{\omega}(\bar{X})\big[\pounds_{\bar{V}}\big(\tilde{\sigma}(\bar{Y})\big) - \tilde{\sigma}(\pounds_{\bar{V}}\bar{Y})\big] \\
&= (\pounds_{\bar{V}}\tilde{\omega})(\bar{X})\tilde{\sigma}(\bar{Y}) + \tilde{\omega}(\bar{X})(\pounds_{\bar{V}}\tilde{\sigma})(\bar{Y}) \\
&= \big((\pounds_{\bar{V}}\tilde{\omega}) \otimes \tilde{\sigma} + \tilde{\omega} \otimes (\pounds_{\bar{V}}\tilde{\sigma})\big)(\bar{X}, \bar{Y})
\end{aligned}$$

Hence

$$\begin{aligned}
\pounds_{\bar{V}}\big(\tilde{\omega} \wedge \tilde{\sigma}) &= (\pounds_{\bar{V}}\tilde{\omega}) \otimes \tilde{\sigma} + \tilde{\omega} \otimes (\pounds_{\bar{V}}\tilde{\sigma}) - (\pounds_{\bar{V}}\tilde{\sigma}) \otimes \tilde{\omega} - \tilde{\sigma} \otimes (\pounds_{\bar{V}}\tilde{\omega}) \\
&= (\pounds_{\bar{V}}\tilde{\omega}) \wedge \tilde{\sigma} + \tilde{\omega} \wedge (\pounds_{\bar{V}}\tilde{\sigma})
\end{aligned}$$

(c) Set $\tilde{\omega} = \tilde{\mathrm{d}}x^i$ and $\tilde{\sigma} = \tilde{\mathrm{d}}x^j$. They have the components $\omega_k = \delta^i{}_k$ and $\sigma_l = \delta^j{}_l$. From the result of part (a), we have

$$(\pounds_{\bar{V}}\tilde{\mathrm{d}}x^i)_k = V^j \partial_j \delta^i{}_k + \delta^i{}_j \partial_k V^j = \partial_k V^i$$

or, equivalently,

$$\pounds_{\bar{V}}\tilde{\mathrm{d}}x^i = \partial_k V^i \, \tilde{\mathrm{d}}x^k$$

and similarly for $\pounds_{\bar{V}}\tilde{\mathrm{d}}x^j$.

From the result of part (b),

$$\begin{aligned}
\pounds_{\bar{V}}\big(\tilde{\mathrm{d}}x^i \wedge \tilde{\mathrm{d}}x^j) &= (\pounds_{\bar{V}}\tilde{\mathrm{d}}x^i) \wedge \tilde{\mathrm{d}}x^j + \tilde{\mathrm{d}}x^i \wedge (\pounds_{\bar{V}}\tilde{\mathrm{d}}x^j) \\
&= \partial_k V^i \, \tilde{\mathrm{d}}x^k \wedge \tilde{\mathrm{d}}x^j + \partial_k V^j \, \tilde{\mathrm{d}}x^i \wedge \tilde{\mathrm{d}}x^k
\end{aligned}$$