NATIONAL UNIVERSITY OF SINGAPORE

PC4274 MATHEMATICAL METHODS IN PHYSICS III

(Semester II: AY 2017–18)

Time Allowed: 2 Hours

INSTRUCTIONS TO STUDENTS

1. Write your matriculation number only. Do not write your name.
2. This examination paper contains **FOUR** questions and comprises **FOUR** printed pages.
3. Answer **ALL** questions.
4. Please start each question on a new page.
5. This is a **CLOSED BOOK** examination.
6. The total mark for this examination paper is 100.

1. Consider the following vector field $\bar{V}$ and one-form field $\tilde{\omega}$:

$$\bar{V} = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\,, \qquad \tilde{\omega} = -t\,\tilde{\mathrm{d}}t + x\,\tilde{\mathrm{d}}x\,,$$

in Minkowski space with coordinates (t, x, y, z). The metric tensor for this space is given by

$$\boldsymbol{g} = -\,\tilde{\mathrm{d}}t \otimes \tilde{\mathrm{d}}t + \tilde{\mathrm{d}}x \otimes \tilde{\mathrm{d}}x + \tilde{\mathrm{d}}y \otimes \tilde{\mathrm{d}}y + \tilde{\mathrm{d}}z \otimes \tilde{\mathrm{d}}z\,.$$

(a) Calculate the following quantities:

(i) $\tilde{\omega}(\bar{V})$; (ii) $\boldsymbol{g}(\bar{V}, \cdot\,)$; (iii) $\boldsymbol{g}(\bar{V}, \bar{V})$.

(b) Find a function $f = f(t, x)$ such that $\tilde{\omega} = \tilde{\mathrm{d}}f$. Hence, or otherwise, sketch the one-form field $\tilde{\omega}$ at the following eight points in the (t, x)-plane: $(\pm 1, 0)$, $(0, \pm 1)$, $(\pm 1, \pm 1)$ and $(\pm 1, \mp 1)$.

[20 marks]

2. Let B_{ij} be components of an antisymmetric $\binom{0}{2}$ tensor in coordinates $\{x^i\}$. Suppose we perform a general coordinate transformation of the form $y^{i'} = y^{i'}(x^i)$, and let $B_{i'j'}$ be components of this tensor in the new coordinates.

(a) Write down an expression for $B_{i'j'}$ in terms of B_{ij}. Hence show that $B_{i'j'}$ is also antisymmetric.

(b) Show that under this coordinate transformation, $\partial_i B_{jk} \equiv \frac{\partial B_{jk}}{\partial x^i}$ transforms as follows:

$$\frac{\partial B_{j'k'}}{\partial y^{i'}} = \frac{\partial x^i}{\partial y^{i'}}\frac{\partial x^j}{\partial y^{j'}}\frac{\partial x^k}{\partial y^{k'}}\frac{\partial B_{jk}}{\partial x^i} + \frac{\partial^2 x^j}{\partial y^{i'}\partial y^{j'}}\frac{\partial x^k}{\partial y^{k'}}B_{jk} + \frac{\partial x^j}{\partial y^{j'}}\frac{\partial^2 x^k}{\partial y^{i'}\partial y^{k'}}B_{jk}\,.$$

Hence explain why $\partial_i B_{jk}$ are **not** components of a $\binom{0}{3}$ tensor.

(c) Now define $H_{ijk} \equiv \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}$. Show that H_{ijk} transform as components of a $\binom{0}{3}$ tensor.

[30 marks]

3. Consider the following one-form field:

$$\tilde{\alpha} = P(x,y,z)\,\tilde{\mathrm{d}}x + Q(x,y,z)\,\tilde{\mathrm{d}}y + R(x,y,z)\,\tilde{\mathrm{d}}z\,,$$

in Euclidean space $\mathbb{R}^3$ with standard Cartesian coordinates (x, y, z).

(a) Show that

$$\tilde{\mathrm{d}}\tilde{\alpha} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\tilde{\mathrm{d}}z \wedge \tilde{\mathrm{d}}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y\,. \tag{1}$$

Further show that $\tilde{\mathrm{d}}^2\tilde{\alpha} = 0$.

(b) By using Stokes' theorem and Eq. (1), write down an expression for the (closed) line integral $\int_C P\,\tilde{\mathrm{d}}x + Q\,\tilde{\mathrm{d}}y + R\,\tilde{\mathrm{d}}z$ in terms of a surface integral. Hence calculate the line integral

$$\int_C z(x^2-1)\,\tilde{\mathrm{d}}x + y(x+1)\,\tilde{\mathrm{d}}z\,,$$

where C is the unit circle $x^2 + y^2 = 1$ in the $z = 0$ plane.

[20 marks]

4. Let M be an n-dimensional manifold with coordinates $\{x^i\}$.

(a) If $\tilde{\omega}$ is a one-form field on M, show that its Lie derivative along a vector field $\bar{V}$ has components

$$(\pounds_{\bar{V}}\tilde{\omega})_i = V^j\partial_j\omega_i + \omega_j\partial_i V^j,$$

in a coordinate basis. (Recall that the Lie derivative of a scalar field and vector field are $\pounds_{\bar{V}} f = \bar{V}f$ and $(\pounds_{\bar{V}}\bar{W})^i = [\bar{V}, \bar{W}]^i = V^j\partial_j W^i - W^j\partial_j V^i$, respectively.)

(Question continued on next page)

(b) If $\tilde{\omega}$ and $\tilde{\sigma}$ are both one-form fields on M, show that

$$\pounds_{\bar{V}}(\tilde{\omega} \wedge \tilde{\sigma}) = (\pounds_{\bar{V}}\tilde{\omega}) \wedge \tilde{\sigma} + \tilde{\omega} \wedge (\pounds_{\bar{V}}\tilde{\sigma})\,.$$

[*Hint: Express the wedge product in terms of the tensor product and then show that* $\pounds_{\bar{V}}(\tilde{\omega} \otimes \tilde{\sigma}) = (\pounds_{\bar{V}}\tilde{\omega}) \otimes \tilde{\sigma} + \tilde{\omega} \otimes (\pounds_{\bar{V}}\tilde{\sigma})$.]

(c) Using the results of parts (a) and (b), evaluate $\pounds_{\bar{V}}(\tilde{\mathrm{d}}x^i \wedge \tilde{\mathrm{d}}x^j)$.

[30 marks]

(ET)

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