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NOTE

Windowing Techniques for Image Restoration

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The large errors observed in inverse filter or Wiener filter restorations of images are mainly due to the fact that only a truncated region of image data is available for processing. In the earlier literature, it was suggested that the well-known time-series windows may be generalized for treating these errors. This paper examines the windowing technique for the restoration of general blurred images. Mathematical expressions for the restoration errors that arise from truncated data are derived. Optimal windows for image restoration are then designed on the basis of these expressions. With these optimal windows, near-perfect restorations can be obtained if the images vary gradually in intensity near their borders. Restorations using the optimal windows and some well-known time-series analysis windows are presented for comparison of their performance.

I. INTRODUCTION

Inverse or Wiener filter restorations of blurred images are often marred by large errors which have often been attributed to the presence of noise. Pratt [1] proposed that truncation of the image data could be a cause of the problem and suggested that the windows designed for the spectral analysis of time series might be used to reduce the restoration error. Woods *et al.* [2] have also examined the boundary value problem in the implementation of a Kalman filter for image restoration. Lim *et al.* [3] presented mathematical expressions which show that the restoration errors for motion-blurred images are proportional to the difference between the pixel intensities in the left and right borders.

Recent research in error reduction has been based on the theory of projections onto convex sets (Lagendijk *et al.* [4]; Sezan and Tekalp [5]). In these new methods, errors are kept in control through the use of image models based on a priori knowledge and constrained minimization techniques. Computation cost is however high as iterative computation is required.

Lim *et al.* [3] designed an optimal window for use in inverse and Wiener filter restoration of motion-blurred images. This window is defined in terms of the point spread function (PSF). It removes the restoration errors due to the difference between the average pixel intensities of the left and right borders. However, variations in pixel intensity within the left or the right border regions will still give rise to some residual errors. This window is readily implemented for general motion-blurred images at negligible computational overhead and is therefore useful for operational restoration work. In this paper, we extend the analysis and design an optimal window for general two-dimensional images blurred by a space-invariant point spread function. Some specific examples of images restored with this optimal window and the standard time-series windows are presented for comparison of their performance.

II. WINDOW FOR GENERAL TWO-DIMENSIONAL BLURRED IMAGES

Suppose $g_{i,k}$ is an $N \times N$ digitized image of a scene $f_{i,k}$ blurred by a two-dimensional PSF $h_{i,k}$,

$$g_{i,k} = \sum_{j=0}^{M-1} \sum_{l=0}^{L-1} h_{j,l} f_{i-j,k-l}, \tag{1}$$

where M and L are the horizontal and vertical extents of the PSF, respectively. By taking the DFT of $g_{i,k}$, we obtain

$$\begin{aligned} G_{u,v} &= \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} W^{iu+kv} \sum_{j=0}^{M-1} \sum_{l=0}^{L-1} h_{j,l} f_{i-j,k-l} \\ &= \sum_{j=0}^{M-1} \sum_{l=0}^{L-1} h_{j,l} W^{ju+lv} \sum_{i=-j}^{N-1-j} \sum_{k=-l}^{N-1-l} f_{i,k} W^{iu+kv} \end{aligned}$$

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$$= H_{u,v}F_{u,v} + \sum_{j=1}^{M-1}\sum_{l=1}^{L-1} h_{j,l}W^{ju+lv}$$

$$\times \Bigg\{ \sum_{k=1}^{l} [G^{(1)}_{u,-k} - G^{(1)}_{u,N-k}] W^{-kv}$$

$$+ \sum_{i=1}^{j} [G^{(2)}_{-i,v} - G^{(2)}_{N-i,v}] W^{-iu}$$

$$+ \sum_{k=1}^{l}\sum_{i=1}^{j} [f_{-i,-k} + f_{N-i,N-k}$$

$$- f_{N-i,-k} - f_{-i,N-k}] W^{-iu-kv} \Bigg\}$$

$$= H_{u,v}F_{u,v} + E_{u,v}, \quad (2)$$

where $G^{(1)}_{u,k} = \sum_{i=0}^{N-1} f_{i,k}W^{iu}$ and $G^{(2)}_{i,v} = \sum_{k=0}^{N-1} f_{i,k}W^{ku}$.

The term $H_{u,v}F_{u,v}$ by itself will give a perfect image when restored with the inverse filter. The term $E_{u,v}$ represents restoration errors that arise due to the lack of periodicity in the image. Following [3], we call this error the edge error. Except for special cases of images which are periodic in both the horizontal and vertical directions, restorations using an inverse or Wiener filter are generally overwhelmed by the edge error.

We design a multiplicative window $\omega_{i,k}$ for the reduction of the edge error. The strategy is as follows. The DFT of the product of the blurred image and the window function $\omega_{i,k}g_{i,k}$ is first expanded as in Eq. (2), but now the edge error ($E_{u,v}$) and other terms are modified by the window function $\omega_{i,k}$. On the basis of the new edge error term in the DFT, the window function $\omega_{i,k}$ may then be determined so as to minimize the expected restoration error. The steps to be taken are similar to those described in [3] for the case of motion-blur images. Unfortunately the equations become rather involved. We therefore relegate the detailed manipulations to the appendices and present below only the main equation for a description of the steps leading to the design. The DFT of the windowed blurred image (i.e., $\omega_{i,k}g_{i,k}$) is given by the expression

$$\tilde{G}_{u,v} = \sum_{j=0}^{M-1}\sum_{l=0}^{L-1} \alpha(j,l,u,v) \sum_{i=0}^{N-M}\sum_{k=0}^{N-L} \beta(i,k,u,v)$$

$$+ \sum_{j=0}^{M-1}\sum_{l=0}^{L-1} \alpha(j,l,u,v) \sum_{i=N-M+1}^{N-1}\sum_{k=0}^{N-L} \beta(i,k,u,v)$$

$$+ \sum_{j=0}^{M-1}\sum_{l=0}^{L-1} \alpha(j,l,u,v) \sum_{i=0}^{N-1}\sum_{k=N-L+1}^{N-1} \beta(i,k,u,v)$$

$$- \sum_{j=0}^{M-1}\sum_{l=0}^{L-1} \alpha(j,l,u,v) \sum_{i=0}^{N-1}\sum_{k=0}^{N-1} (1 - \omega_{i+j,k+l})\beta(i,k,u,v)$$

$$+ \sum_{j=0}^{M-1}\sum_{l=0}^{L-1} \alpha(j,l,u,v) \sum_{i=0}^{N-1}\sum_{k=-l}^{-1} \gamma(i,j,k,l,u,v)$$

$$- \sum_{j=0}^{M-1}\sum_{l=1}^{L-1} \alpha(j,l,u,v) \sum_{i=0}^{N-1}\sum_{k=N-l}^{N-1} \gamma(i,j,k,l,u,v)$$

$$+ \sum_{j=1}^{M-1}\sum_{l=0}^{L-1} \alpha(j,l,u,v) \sum_{i=-j}^{-1}\sum_{k=0}^{N-1} \gamma(i,j,k,l,u,v)$$

$$+ \sum_{j=1}^{M-1}\sum_{l=1}^{L-1} \alpha(j,l,u,v) \sum_{i=-j}^{-1}\sum_{k=-l}^{-1} \gamma(i,j,k,l,u,v)$$

$$- \sum_{j=1}^{M-1}\sum_{l=1}^{L-1} \alpha(j,l,u,v) \sum_{i=-j}^{-1}\sum_{k=N-l}^{N-1} \gamma(i,j,k,l,u,v)$$

$$- \sum_{j=1}^{M-1}\sum_{l=0}^{L-1} \alpha(j,l,u,v) \sum_{i=N-j}^{N-1}\sum_{k=0}^{N-1} \gamma(i,j,k,l,u,v)$$

$$- \sum_{j=1}^{M-1}\sum_{l=1}^{L-1} \alpha(j,l,u,v) \sum_{i=N-j}^{N-1}\sum_{k=-l}^{-1} \gamma(i,j,k,l,u,v)$$

$$+ \sum_{j=1}^{M-1}\sum_{l=1}^{L-1} \alpha(j,l,u,v) \sum_{i=N-j}^{N-1}\sum_{k=N-l}^{N-1} \gamma(i,j,k,l,u,v)$$

$$= HF^{(1)}_{u,v} + HF^{(2)}_{u,v} + W_{u,v} + E_{u,v},$$

where

$$\alpha(j,l,u,v) = h_{j,l}W^{ju+lv},$$

$$\beta(i,k,u,v) = f_{i,k}W^{iu+kv},$$

$$\gamma(i,j,k,l,u,v) = \omega_{i+j,k+l}\beta(i,k,u,v).$$

$HF^{(1)}_{u,v}$ denotes Terms 1 and $HF^{(2)}_{u,v}$ denotes the sum of Terms 2 and 3 of (3). An inverse filtering of these two terms ($HF^{(1)}_{u,v} + HF^{(2)}_{u,v}$) would give a perfect restoration. They are however written here as two separate terms to aid us in the definition of a suitable window. Analogous to the one-dimensional case presented in [3], $HF^{(2)}_{u,v}$ represents the narrow L-shape border at the right and bottom edges of the image, which we have to sacrifice for reduction of restoration errors over the rest of the image. $W_{u,v}$ denotes Term 4 of (3), which represents the windowing error. It vanishes when no windowing is applied ($\omega_{i,k} = 1$). $E_{u,v}$ denotes Terms 5–12, which represent the edge error arising from the nonperiodicity of the image.

To determine $\omega_{i,k}$, we rewrite $HF^{(2)}_{u,v}$, $W_{u,v}$, and $E_{u,v}$ as summations with respect to the subscripts of $\omega_{i,k}$. The various summations are broken into terms which involve $\omega_{i,k}$ in each of the nine regions of the image as shown in Fig. 1. The details are presented in Appendices A, B, and C, respectively. In the central region of Fig. 1, we set $\omega_{i,k} = 1$. In Appendix D, $\omega_{i,k}$ are determined for each of the eight remaining regions. The optimal window $\omega_{i,k}$ as determined is summarized as

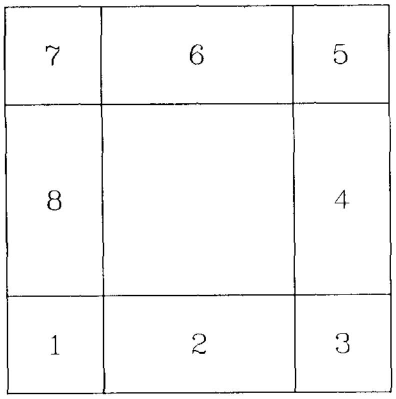

FIG. 1. The image plane is divided into nine regions in the design of a restoration window ω_{ik}. In the unnumbered central region $\omega_{i,k} = 1$. The values for $\omega_{i,k}$ in Regions 1–8 are derived in Appendix D and summarized in Eq. (4). The image plane is divided as follows: Regions 1, 7, and 8 have $0 \le i \le (M-2)$, Regions 2 and 6 and the unnumbered central region have $(M-1) \le i \le (N-M)$, and Regions 3, 4, and 5 have $(N-M+1) \le i \le (N-1)$. In the other direction, Regions 1, 2, and 3 have $0 \le k \le (L-2)$, Regions 4 and 8 and the unnumbered central region have $(L-1) \le k \le (N-L)$, and Regions 5, 6, and 7 have $(N-L+1) \le k \le (N-1)$.

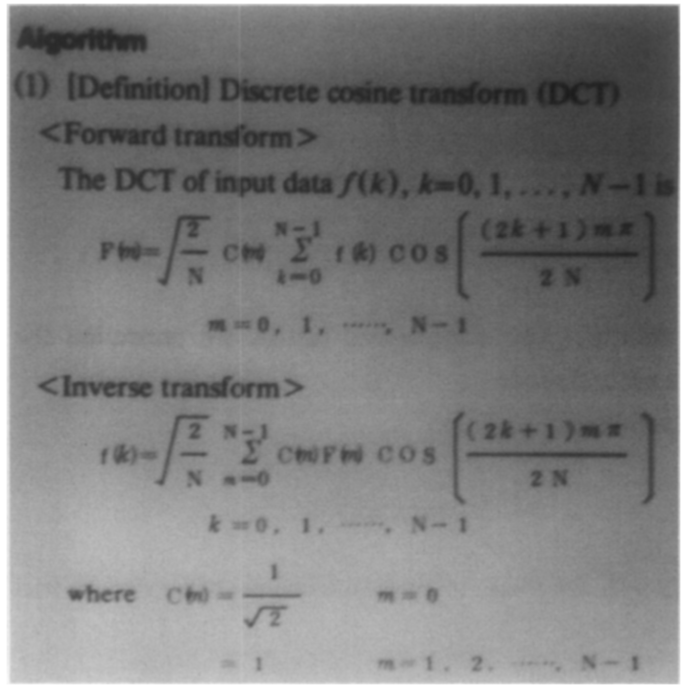
Algorithm
(1) [Definition] Discrete cosine transform (DCT)
<Forward transform>
The DCT of input data $f(k)$, $k=0, 1, \ldots, N-1$ is
$$F(m) = \sqrt{\frac{2}{N}}\, C(m) \sum_{k=0}^{N-1} f(k) \cos\left[\frac{(2k+1)m\pi}{2N}\right]$$
$m = 0, 1, \cdots, N-1$
<Inverse transform>
$$f(k) = \sqrt{\frac{2}{N}} \sum_{m=0}^{N-1} C(m) F(m) \cos\left[\frac{(2k+1)m\pi}{2N}\right]$$
$k = 0, 1, \cdots, N-1$
where $C(m) = \frac{1}{\sqrt{2}}$ $m = 0$
$= 1$ $m = 1, 2, \cdots, N-1$

FIG. 2. The original image used for the restoration experiment.

$$\begin{pmatrix} \sum_{p=0}^{i} \sum_{q=k+L-N}^{L-1} h_{p,q} & \sum_{p=0}^{M-1} \sum_{q=k+L-N}^{L-1} h_{p,q} & \sum_{p=i+M-N}^{M-1} \sum_{q=k+L-N}^{L-1} h_{p,q} \\ \sum_{p=0}^{i} \sum_{q=0}^{L-1} h_{p,q} & 1 & \sum_{p=i+M-N}^{M-1} \sum_{q=0}^{L-1} h_{p,q} \\ \sum_{p=0}^{i} \sum_{q=0}^{k} h_{p,q} & \sum_{p=0}^{M-1} \sum_{q=0}^{k} h_{p,q} & \sum_{p=i+M-N}^{M-1} \sum_{q=0}^{k} h_{p,q} \end{pmatrix}, \quad (4)$$

where the entries in the large matrix apply to the corresponding regions of Fig. 1. Window (4) will eliminate restoration errors due to the differences in the average intensities between the left and right edges and between the top and bottom edges. However, a residual error will still remain if the pixel intensity departs from uniformity in the regions near the border.

III. COMPARISON WITH TIME-SERIES WINDOWS

In this section, we present some examples of images restored using a Wiener filter together with windowing. The DFTs $\tilde{G}_{u,v}$ of the windowed blurred images $\omega_{i,k} g_{i,k}$ are first obtained and the restored images are then computed by the inverse DFT of $\tilde{G}_{u,v} H^*_{u,v}/(|H_{u,v}|^2 + \gamma)$. On the basis of the recommendation of [3], we have chosen $\gamma = 0.0001$ for all the examples shown.

The original sharp image of a page of text is shown in Fig. 2. Figure 3 shows the blurred image obtained by convolving Fig. 2 with a cylindrical PSF, i.e., $h_{i,k} = 1$ for $i^2 + k^2 \le r^2$ and $h_{i,k} = 0$ otherwise, where $r = 8$. The Wiener filter restoration obtained without windowing is shown in Fig. 4. As expected, the restoration is dominated by edge errors.

Figures 5a and 5b show the restored images obtained using the Bartlett window; in (a) the two-dimensional window is generated by rotating the time-series window through 2π and in (b) it is generated by multiplying the two corresponding one-dimensional windows, one for each direction. The restored image retained only a small portion of the original image near the center. The edge error which showed up as horizontal and vertical strips in Fig. 4 is now suppressed. There is however a loss of image intensity away from the center, which is a manifestation of the windowing error. Restorations with the Blackman and Hamming windows have also been attempted. The results are similar except that the Blackman window restorations show even stronger windowing error while the Hamming window restorations show visible edge error. All the restorations are not satisfactory for practical use as the windowing errors are too severe, especially for the rotated windows.

Figure 6 shows the restoration obtained using the window defined by (4). We observe that the edge error has been effectively controlled and the windowing error consists of only the loss of a narrow L-shape border. The

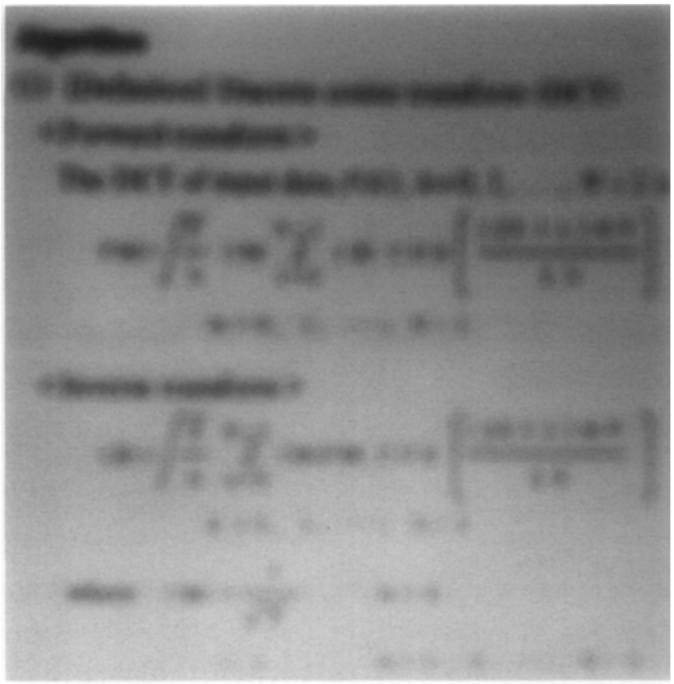

FIG. 3. The image of Fig. 2 blurred by a cylindrical PSF.

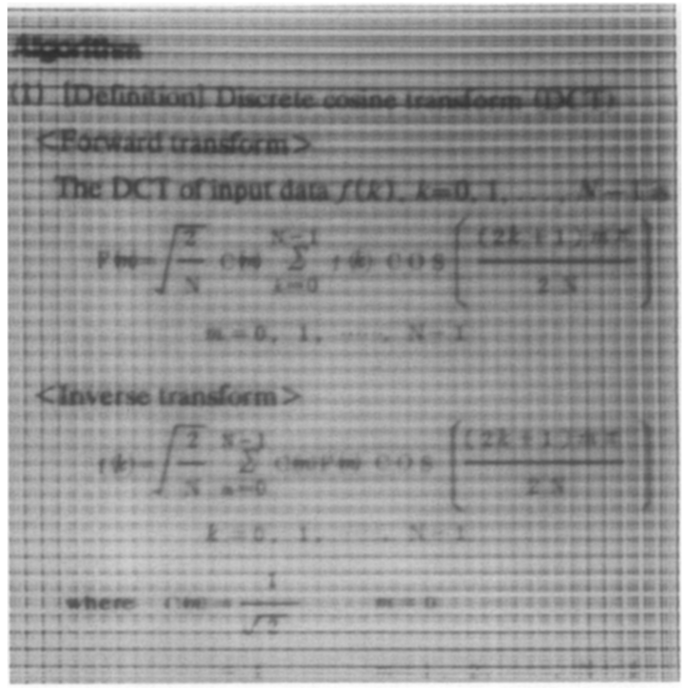

FIG. 4. Wiener filter restoration of Fig. 2 without windowing. Edge error appears as periodic horizontal and vertical lines.

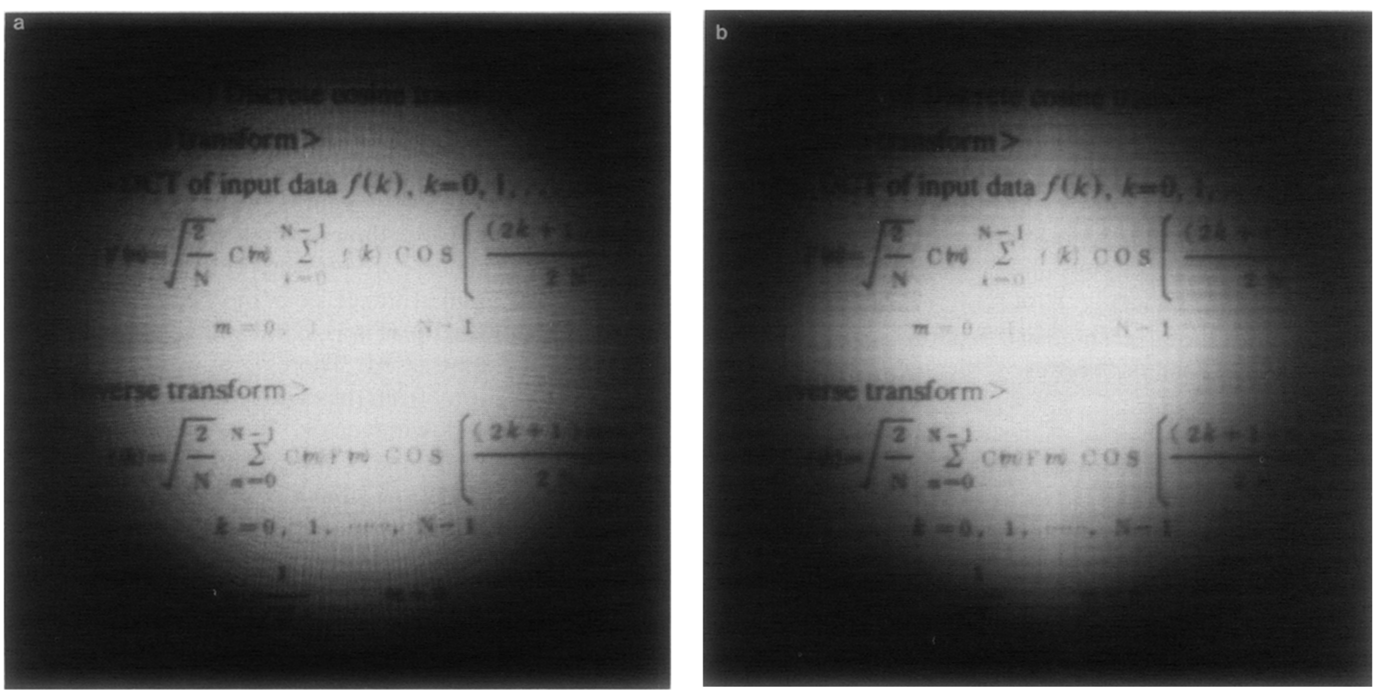

FIG. 5. Wiener filter restoration of Fig. 2 with Bartlett window. (a) The 2D window is obtained by rotating the traditional Bartlett window. (b) The 2D window is obtained by multiplying two traditional Bartlett windows, one for each direction. Although the edge error is reduced, much of the image is lost due to windowing error.

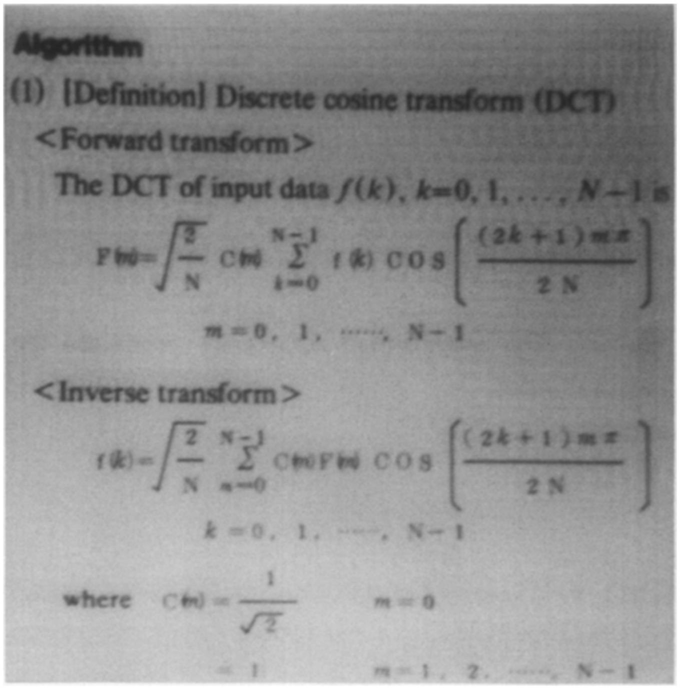

FIG. 6. Wiener filter restoration of Fig. 2 with window (4). Residual edge error is not significantly stronger than those observed at the central regions of Fig. 5. Windowing error is minimized to an L-shape border on the right and the bottom of the image.

superiority of this window over the time-series windows for image restoration work is therefore clearly demonstrated.

IV. SUMMARY AND CONCLUSIONS

We have thus shown that windowing techniques similar to those used in the spectral analysis of time series by DFT techniques may also be applied to digital image restoration. However, the windows designed for time-series analysis are not optimal for this purpose since they have been specifically designed for the reduction of spectral leakage.

We have obtained the exact expressions for the edge error and windowing error and have derived from these expressions optimal windows that minimize the restoration errors for general motion or defocusing blurred images. These windows are very effective in reducing the edge error of images with pixel intensity varying gradually along their borders. These windows also limit the windowing error to the loss of a narrow strip. In comparison, the time-series windows all suffer from very strong windowing error although they generally effectively control the edge error.

APPENDIX A: $HF^{(2)}_{u,v}$ TERMS OF EQ. (3)

The terms denoted by $HF^{(2)}_{u,v}$ in Eq. (3) can be split into 14 terms as

$$\sum_{j=0}^{M-1}\sum_{l=0}^{L-1} h_{j,l} W^{ju+lv}\left(\sum_{i=N-M+1}^{N-1}\sum_{k=0}^{N-L} f_{i,k} W^{iu+kv} + \sum_{i=0}^{N-1}\sum_{k=N-L+1}^{N-1} f_{i,k} W^{iu+kv}\right)$$

$$= \begin{bmatrix} N-1 & L-2 & p & q \\ N-M+1 & 0 & N-M+1 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} N+M-2 & L-2 & N-1 & q \\ N & 0 & p-M+1 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} N-1 & N-L & p & q \\ N-M+1 & L-1 & N-M+1 & q-L+1 \end{bmatrix}$$

$$+ \begin{bmatrix} N+M-2 & N-L & N-1 & q \\ N & L-1 & p-M+1 & q-L+1 \end{bmatrix}$$

$$+ \begin{bmatrix} N-1 & N-1 & p & N-L \\ N-M+1 & N-L-1 & N-M+1 & q-L+1 \end{bmatrix}$$

$$+ \begin{bmatrix} N+M-2 & N-L & N-1 & N-L \\ N & N-L+1 & p-M+1 & q-L+1 \end{bmatrix}$$

$$+\begin{bmatrix} M-2 & N-L & p & q \\ 0 & N-L+1 & 0 & N-L+1 \end{bmatrix} \tag{A1}$$

$$+\begin{bmatrix} M-2 & N+L-2 & p & N-1 \\ 0 & N & 0 & q-L+1 \end{bmatrix}$$

$$+\begin{bmatrix} N-1 & N+L-2 & p & N-L \\ N-M+1 & N & p-M+1 & q-L+1 \end{bmatrix}$$

$$+\begin{bmatrix} N-M & N+L-2 & p & N-L \\ M-1 & N & p-M+1 & q-L+1 \end{bmatrix}$$

$$+\begin{bmatrix} N-1 & N-1 & p & q \\ N-M+1 & N-L+1 & p-M+1 & N-L+1 \end{bmatrix}$$

$$+\begin{bmatrix} N-M & N-L & p & q \\ M-1 & N-L+1 & p-M+1 & N-L+1 \end{bmatrix}$$

$$+\begin{bmatrix} N+M-2 & N-L & N-1 & q \\ N & N-L+1 & p-M+1 & N-L+1 \end{bmatrix}$$

$$+\begin{bmatrix} N+M-2 & N+L-2 & N-1 & N-1 \\ N & N & p-M+1 & q-L+1 \end{bmatrix},$$

where

$$\begin{bmatrix} b & d & y & z \\ a & c & x & w \end{bmatrix} = \sum_{p=a}^{b} \sum_{q=c}^{d} W^{pu+qv} \sum_{i=x}^{y} \sum_{k=w}^{z} h_{p-i,q-k} f_{i,k}.$$

APPENDIX B: WINDOWING ERROR TERM $W_{u,v}$ OF EQ. (3)

The windowing error term denoted by $W_{u,v}$ in Eq. (3) can be split into 16 terms as

$$-\sum_{j=0}^{M-1} \sum_{l=0}^{L-1} h_{j,l} W^{ju+lv} \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} (1-\omega_{i+j,k+l}) f_{i,k} W^{iu+kv}$$

$$= -\begin{Bmatrix} M-2 & L-2 & p & q \\ 0 & 0 & 0 & 0 \end{Bmatrix} - \begin{Bmatrix} M-2 & N-1 & p & q \\ 0 & N-L+1 & 0 & q-L+1 \end{Bmatrix}$$

$$-\begin{Bmatrix} M-2 & N+L-2 & p & N-1 \\ 0 & N & 0 & q-L+1 \end{Bmatrix} - \begin{Bmatrix} M-2 & N-L & p & q \\ 0 & L-1 & 0 & q-L+1 \end{Bmatrix}$$

$$-\begin{Bmatrix} N-1 & L-2 & p & q \\ N-M+1 & 0 & p-M+1 & 0 \end{Bmatrix}$$

$$-\begin{Bmatrix} N-1 & N-1 & p & q \\ N-M+1 & N-L+1 & p-M+1 & q-L+1 \end{Bmatrix}$$

$$-\begin{Bmatrix} N-1 & N+L-2 & p & N-1 \\ N-M+1 & N & p-M+1 & q-L+1 \end{Bmatrix}$$

$$-\begin{Bmatrix} N-1 & N-L & p & q \\ N-M+1 & L-1 & p-M+1 & q-L+1 \end{Bmatrix}$$

$$-\begin{Bmatrix} N+M-2 & L-2 & N-1 & q \\ N & 0 & p-M+1 & 0 \end{Bmatrix} \tag{B1}$$

$$-\begin{Bmatrix} N+M-2 & N-1 & N-1 & q \\ N & N-L+1 & p-M+1 & q-L+1 \end{Bmatrix}$$

$$- \begin{Bmatrix} N+M-2 & N+L-2 & N-1 & N-1 \\ N & N & p-M+1 & q-L+1 \end{Bmatrix}$$

$$- \begin{Bmatrix} N+M-2 & N-L & N-1 & q \\ N & L-1 & p-M+1 & q-L+1 \end{Bmatrix}$$

$$- \begin{Bmatrix} N-M & L-2 & p & q \\ M-1 & 0 & p-M+1 & 0 \end{Bmatrix}$$

$$- \begin{Bmatrix} N-M & N-1 & p & q \\ M-1 & N-L+1 & p-M+1 & q-L+1 \end{Bmatrix}$$

$$- \begin{Bmatrix} N-M & N+L-2 & p & N-1 \\ M-1 & N & p-M+1 & q-L+1 \end{Bmatrix}$$

$$- \begin{Bmatrix} N-M & N-L & p & q \\ M-1 & L-1 & p-M+1 & q-L+1 \end{Bmatrix},$$

where

$$\begin{Bmatrix} b & d & y & z \\ a & c & x & w \end{Bmatrix} = \sum_{p=a}^{b} \sum_{q=c}^{d} (1 - \omega_{p,q}) W^{pu+qv} \sum_{i=x}^{y} \sum_{k=w}^{z} h_{p-i,q-k} f_{i,k}.$$

APPENDIX C: EDGE ERROR TERM $E_{u,v}$ OF EQ. (3)

$E_{u,v}$ represents Terms 5–8 of Eq. (3). Each of them are transformed as shown below, where $\xi(p, q, u, v)$ denotes the product $W^{pu+qv}\omega_{p,q}$.

Term 5:

$$\sum_{j=0}^{M-1} \sum_{l=1}^{L-1} h_{j,l} W^{ju+lv} \sum_{i=0}^{N-1} \sum_{k=-l}^{-1} \omega_{i+j,k+l} f_{i,k} W^{iu+kv}$$
$$= \sum_{p=0}^{M-2} \sum_{q=0}^{L-2} \xi(p, q, u, v) \sum_{i=0}^{p} \sum_{k=1}^{L-q-1} \eta(i, k, p, q)$$
$$+ \sum_{p=N-M+1}^{N-1} \sum_{q=0}^{L-2} \xi(p, q, u, v) \sum_{i=p-M+1}^{p} \sum_{k=1}^{L-q-1} \eta(i, k, p, q)$$
$$+ \sum_{p=N}^{N+M-2} \sum_{q=0}^{L-2} \xi(p, q, u, v) \sum_{i=p-M+1}^{N-1} \sum_{k=1}^{L-q-1} \eta(i, k, p, q)$$
$$+ \sum_{p=M-1}^{N-M} \sum_{q=0}^{L-2} \xi(p, q, u, v) \sum_{i=p-M+1}^{p} \sum_{k=1}^{L-q-1} \eta(i, k, p, q), \tag{C1}$$

where $\eta(i, k, p, q) = f_{i,-k} h_{p-i,q+k}$.

Term 6:

$$- \sum_{j=0}^{M-1} \sum_{l=1}^{L-1} h_{j,l} W^{ju+lv} \sum_{i=0}^{N-1} \sum_{k=N-l}^{N-1} \omega_{i+j,k+l} f_{i,k} W^{iu+kv}$$
$$= - \sum_{p=0}^{M-2} \sum_{q=0}^{L-2} \xi(p, q, u, v) \sum_{i=0}^{p} \sum_{k=1}^{L-q-1} \mu(i, k, p, q)$$
$$- \sum_{p=N-M+1}^{N-1} \sum_{q=0}^{L-2} \xi(p, q, u, v) \sum_{i=p-M+1}^{p} \sum_{k=1}^{L-q-1} \mu(i, k, p, q)$$
$$- \sum_{p=N}^{N+M-2} \sum_{q=0}^{L-2} \xi(p, q, u, v) \sum_{i=p-M+1}^{N-1} \sum_{k=1}^{L-q-1} \mu(i, k, p, q)$$
$$- \sum_{p=M-1}^{N-M} \sum_{q=0}^{L-2} \xi(p, q, u, v) \sum_{i=p-M+1}^{p} \sum_{k=1}^{L-q-1} \mu(i, k, p, q), \tag{C2}$$

where $\mu(i, k, p, q) = f_{i,N-k} h_{p-i,q+k}$.

Term 7:

$$\sum_{j=1}^{M-1} \sum_{l=0}^{L-1} h_{j,l} W^{ju+lv} \sum_{i=-j}^{-1} \sum_{k=0}^{N-1} \omega_{i+j,k+l} f_{i,k} W^{iu+kv}$$
$$= \sum_{p=0}^{M-2} \sum_{q=0}^{L-2} \xi(p, q, u, v) \sum_{i=1}^{M-p-1} \sum_{k=0}^{q} \delta(i, k, p, q)$$
$$+ \sum_{p=0}^{M-2} \sum_{q=N-L+1}^{N-1} \xi(p, q, u, v) \sum_{i=1}^{M-p-1} \sum_{k=q-L+1}^{q} \delta(i, k, p, q)$$
$$+ \sum_{p=0}^{M-2} \sum_{q=N}^{N+L-2} \xi(p, q, u, v) \sum_{i=1}^{M-p-1} \sum_{k=q-L+1}^{N-1} \delta(i, k, p, q)$$
$$+ \sum_{p=0}^{M-2} \sum_{q=L-1}^{N-L} \xi(p, q, u, v) \sum_{i=1}^{M-p-1} \sum_{k=q-L+1}^{q} \delta(i, k, p, q), \tag{C3}$$

where $\delta(i, k, p, q) = f_{-i,k} h_{p+i,q-k}$.

Term 8:

$$\sum_{j=1}^{M-1}\sum_{l=1}^{L-1} h_{j,l}W^{ju+lv}\sum_{i=-j}^{-1}\sum_{k=-l}^{-1}\omega_{i+j,k+l}f_{i,k}W^{iu+kv} = \sum_{p=0}^{M-2}\sum_{q=0}^{L-2}\xi(p,q,u,v)\sum_{i=1}^{M-p-1}\sum_{k=1}^{L-q-1}f_{-i,-k}h_{p+i,q+k}. \tag{C4}$$

Term 9:

$$-\sum_{j=1}^{M-1}\sum_{l=1}^{L-1} h_{j,l}W^{ju+lv}\sum_{i=-j}^{-1}\sum_{k=N-l}^{N-1}\omega_{i+j,k+l}f_{i,k}W^{iu+kv} = -\sum_{p=0}^{M-2}\sum_{q=0}^{L-2}\xi(p,q,u,v)\sum_{i=1}^{M-p-1}\sum_{k=1}^{L-q-1}f_{-i,N-k}h_{p+i,q+k}. \tag{C5}$$

Term 10:

$$\begin{aligned}-\sum_{j=1}^{M-1}\sum_{l=0}^{L-1} h_{j,l}W^{ju+lv}\sum_{i=N-j}^{N-1}\sum_{k=0}^{N-1}\omega_{i+j,k+l}f_{i,k}W^{iu+kv} &= -\sum_{p=0}^{M-2}\sum_{q=0}^{L-2}\xi(p,q,u,v)\sum_{i=1}^{M-p-1}\sum_{k=0}^{q}\zeta(i,k,p,q)\\ &-\sum_{p=0}^{M-2}\sum_{q=N-L+1}^{N-1}\xi(p,q,u,v)\sum_{i=1}^{M-p-1}\sum_{k=q-L+1}^{q}\zeta(i,k,p,q)\\ &-\sum_{p=0}^{M-2}\sum_{q=N}^{N+L-2}\xi(p,q,u,v)\sum_{i=1}^{M-p-1}\sum_{k=q-L+1}^{N-1}\zeta(i,k,p,q)\\ &-\sum_{p=0}^{M-2}\sum_{q=L-1}^{N-L}\xi(p,q,u,v)\sum_{i=1}^{M-p-1}\sum_{k=q-L+1}^{q}\zeta(i,k,p,q),\end{aligned} \tag{C6}$$

where $\zeta(i,k,p,q) = f_{N-i,k}h_{p+i,q-k}$.

Term 11:

$$-\sum_{j=1}^{M-1}\sum_{l=1}^{L-1} h_{j,l}W^{ju+lv}\sum_{i=N-j}^{N-1}\sum_{k=-l}^{-1}\omega_{i+j,k+l}f_{i,k}W^{iu+kv} = -\sum_{p=0}^{M-2}\sum_{q=0}^{L-2}\xi(p,q,u,v)\sum_{i=1}^{M-p-1}\sum_{k=1}^{L-q-1}f_{N-i,-k}h_{p+i,q+k}. \tag{C7}$$

Term 12:

$$\sum_{j=1}^{M-1}\sum_{l=1}^{L-1} h_{j,l}W^{ju+lv}\sum_{i=N-j}^{N-1}\sum_{k=N-l}^{N-1}\omega_{i+j,k+l}f_{i,k}W^{iu+kv} = \sum_{p=0}^{M-2}\sum_{q=0}^{L-2}\xi(p,q,u,v)\sum_{i=1}^{M-p-1}\sum_{k=1}^{L-q-1}f_{N-i,N-k}h_{p+i,q+k}. \tag{C8}$$

APPENDIX D: DETERMINATION OF $\omega_{p,q}$

To determine the window $\omega_{p,q}$, we put together all the terms of $HF^{(2)}_{u,v}$, $W_{u,v}$, and $E_{u,v}$ listed in Appendices A, B, and C, respectively. We note that each of the following combinations of terms is cancelled out for any choice of $\omega_{p,q}$:

(1) the 8th term of (A1), the 3rd term of (B1), the 1st term of (C2), the 3rd term of (C3), and (C5);

(2) the 2nd term of (A1), the 9th term of (B1), the 3rd term of (C1), the 1st term of (C6), and (C7);

(3) the 14th term of (A1), the 11th term of (B1), the 3rd term of (C2), the 3rd term of (C6), and (C8).

Furthermore, for the central part of the image plane, $(M-1) \le p \le (N-M)$ and $(L-1) \le q \le (N-L)$, we set $\omega_{p,q} = 1$. Term 16 of (B1) therefore also vanishes. The remaining terms contribute to the restoration error. We examine these terms in eight groups, each corresponding to one of the eight border regions of Fig. 1.

• Region 1 $(0 \le p \le M-2,\ 0 \le q \le L-2)$: $\omega_{p,q}$'s which fall within this region are found in the first term of (B1), the first term of (C1), the first term of (C3), and (C4). Adding these terms together we obtain

$$\sum_{p=0}^{M-2}\sum_{q=0}^{L-2}W^{pu+qv}\Bigg\{\omega_{p,q}\Bigg(\sum_{i=0}^{p}\sum_{k=1}^{L-q-1}f_{i,-k}h_{p-i,q+k} + \sum_{i=1}^{M-p-1}\sum_{k=0}^{q}f_{-i,k}h_{p+i,q-k} + \sum_{i=1}^{M-p-1}\sum_{k=1}^{L-q-1}f_{-i,-k}h_{p+i,q+k}\Bigg) - (1-\omega_{p,q})\sum_{i=0}^{p}\sum_{k=0}^{q}f_{i,k}h_{p-i,q-k}\Bigg\}.$$

In general, the above sum will not reduce to zero. For given values of $f_{i,k}$ in and around Region 1, we can define $\omega_{p,q}$ in Region 1 so that the above reduces to zero. However, to design a window which is applicable to general images, we make the reasonable assumption that $f_{i,k}$ varies slowly within the regions $-(M-1) \le i \le (M-2)$ and $-(L-1) \le k \le (L-2)$. In that case, the $f_{i,k}$ terms may be factored out. The condition that the above expression vanishes is then

$$\omega_{p,q}\Bigg(\sum_{i=0}^{p}\sum_{k=0}^{L-1}h_{p-i,k} + \sum_{i=1}^{M-p-1}\sum_{k=0}^{L-1}h_{p+i,k}\Bigg) = \sum_{i=0}^{p}\sum_{k=0}^{q}h_{p-i,q-k}.$$

Since by definition $\sum_{i=0}^{M-1}\sum_{k=0}^{L-1}h_{i,k} = 1$, we have finally

$$\omega_{p,q} = \sum_{i=0}^{p} \sum_{k=0}^{q} h_{i,k}. \tag{D1}$$

• Region 2 $(M - 1 \le p \le N - M, 0 \le q \le L - 2)$; $\omega_{p,q}$'s which fall within this region are found in the 10th term of (A1), the 13th and 15th terms of (B1), the 4th term of (C1), and the 4th term of (C2). Adding these terms together we obtain

$$\sum_{p=M-1}^{N-M} \sum_{q=0}^{L-2} W^{pu+qv} \left\{ \omega_{p,q} \sum_{i=p-M+1}^{p} \sum_{k=1}^{L-q-1} f_{i,k} h_{p-i,q+k} - (1 - \omega_{p,q}) \sum_{i=p-M+1}^{p} \sum_{k=0}^{q} f_{i,k} h_{p-i,q-k} \right\}.$$

If $f_{i,k}$ varies slowly within regions $(p - M + 1) \le i \le p$ and $0 \le k \le (L - 1)$, we have the following condition for the vanishing of the above expression,

$$\omega_{p,q} \left\{ \sum_{i=p-M+1}^{p} \sum_{k=1}^{L-q-1} h_{p-i,q+k} + \sum_{i=p-M+1}^{p} \sum_{k=0}^{q} h_{p-i,q-k} \right\} = \sum_{i=p-M+1}^{p} \sum_{k=0}^{q} h_{p-i,q-k},$$

which reduces to

$$\omega_{p,q} = \sum_{i=0}^{M-1} \sum_{k=0}^{q} h_{i,k}. \tag{D2}$$

• Region 3 $(N - M + 1 \le p \le N - 1, 0 \le q \le L - 2)$: $\omega_{p,q}$'s which fall within this region are found in the first and ninth terms of (A1), the fifth and seventh terms of (B1), the second term of (C1), and the second term of (C2). Adding these terms together we obtain

$$- \sum_{p=N-M+1}^{N-1} \sum_{q=0}^{L-2} W^{pu+qv} \sum_{i=p-M+1}^{N-M} \sum_{k=0}^{q} f_{i,k} h_{p-i,q-k} + \sum_{p=N-M+1}^{N-1} \sum_{q=0}^{L-2} W^{pu+qv} \omega_{p,q} \times \sum_{i=p-M+1}^{p} \left(\sum_{k=1}^{L-q-1} f_{i,-k} h_{p-i,q+k} + \sum_{k=0}^{q} f_{i,k} h_{p-i,q-k} \right).$$

If $f_{i,k}$ varies slowly within regions $(-M + p + 1) \le i \le p$ and $(-L + q + 1) \le k \le q$, the condition for the vanishing of the above expression is

$$\omega_{p,q} \sum_{i=p-M+1}^{p} \left\{ \sum_{k=1}^{L-q-1} h_{p-i,q+k} + \sum_{k=0}^{q} h_{p-i,q-k} \right\} - \sum_{i=p-M+1}^{N-M} \sum_{k=0}^{q} h_{p-i,q-k} = 0,$$

which reduces to

$$\omega_{p,q} = \sum_{i=p+M-N}^{M-1} \sum_{k=0}^{q} h_{i,k}. \tag{D3}$$

• Region 4 $(N - M + 1 \le p \le N - 1, L - 1 \le q \le N - L)$: $\omega_{p,q}$'s which fall within this region are found in the third term of (A1) and the eighth term of (B1). Adding these terms together we obtain

$$- \sum_{p=N-M+1}^{N-1} \sum_{q=L-1}^{N-L} W^{pu+qv} (1 - \omega_{p,q}) \times \sum_{i=p-M+1}^{p} \sum_{k=q-L+1}^{q} f_{i,k} h_{p-i,q-k} + \sum_{p=N-M+1}^{N-1} \sum_{q=L-1}^{N-L} W^{pu+qv} \times \sum_{i=N-M+1}^{p} \sum_{k=q-L+1}^{q} f_{i,k} h_{p-i,q-k}.$$

If $f_{i,k}$ varies slowly within regions $(-M + p + 1) \le i \le p$ and $(-L + q + 1) \le k \le q$, the condition for the vanishing of the above expression is

$$\omega_{p,q} = \sum_{i=p+M-N}^{M-1} \sum_{k=0}^{L-1} h_{i,k}. \tag{D4}$$

• Region 5 $(N - M + 1 \le p \le N - 1, N - L + 1 \le q \le N - 1)$: $\omega_{p,q}$'s which fall within this region are found in the 5th and 11th terms of (A1) and the 6th term of (B1). Adding these terms together we obtain

$$\sum_{p=N-M+1}^{N-1} \sum_{q=N-L+1}^{N-L} W^{pu+qv} \times \left[-(1 - \omega_{p,q}) \sum_{i=p-M+1}^{p} \sum_{k=q-L+1}^{q} f_{i,k} h_{p-i,q-k} + \sum_{i=p-M+1}^{p} \sum_{k=N-L+1}^{q} f_{i,k} h_{p-i,q-k} + \sum_{i=N-M+1}^{p} \sum_{k=q-L+1}^{N-L} f_{i,k} h_{p-i,q-k} \right].$$

If $f_{i,k}$ varies slowly within regions $(N - M + 1) \le i \le N - 1$ and $(N - L + 1) \le k \le N - 1$, the condition for the vanishing of the above expression is

$$\omega_{p,q} = \sum_{i=p+M-N}^{M-1} \sum_{k=q+L-N}^{L-1} h_{i,k}. \quad \text{(D5)}$$

• Region 6 $(M - 1 \le p \le N - M, N - L + 1 \le q \le N - 1)$: $\omega_{p,q}$'s which fall within this region are found in the 12th term of (A1) and the 14th term of (B1). Adding these terms together we obtain

$$- \sum_{p=M-1}^{N-M} \sum_{q=N-L+1}^{N-1} W^{pu+qv}(1 - \omega_{p,q}) \times \sum_{i=p-M+1}^{p} \sum_{k=q-L+1}^{q} f_{i,k} h_{p-i,q-k} + \sum_{p=M-1}^{N-M} \sum_{q=N-L+1}^{N-1} W^{pu+qv} \times \sum_{i=p-M+1}^{p} \sum_{k=N-L+1}^{q} f_{i,k} h_{p-i,q-k}.$$

If $f_{i,k}$ varies slowly within regions $(-M + p + 1) \le i \le p$ and $(-L + q + 1) \le k \le q$, the condition for the vanishing of the above expression is

$$\omega_{p,q} = \sum_{i=0}^{M-1} \sum_{k=q+L-N}^{L-1} h_{i,k}. \quad \text{(D6)}$$

• Region 7 $(0 \le p \le M - 2, N - L + 1 \le q \le N - 1)$: $\omega_{p,q}$'s which fall within this region are found in the 6th, 7th, and 13th terms of (A1), the 2nd and 10th terms of (B1), the 2nd term of (C3), and the 2nd term of (C6). Adding these terms together we obtain

$$\sum_{p=0}^{M-2} \sum_{q=N-L+1}^{N-L} W^{pu+qv} \times \left\{ \omega_{p,q} \sum_{i=1}^{M-p-1} \sum_{k=q-L+1}^{q} f_{-i,k} h_{p+i,q-k} - (1 - \omega_{p,q}) \sum_{i=0}^{p} \sum_{k=q-L+1}^{q} f_{i,k} h_{p-i,q-k} + \sum_{i=0}^{p} \sum_{k=N-L+1}^{q} f_{i,k} h_{p-i,q-k} \right\}.$$

If $f_{i,k}$ varies slowly within the region $(-M + 1) \le i \le (M - 2)$ and $(N - 2L + 2) \le k \le (N - 1)$, the condition for the vanishing of the above expression is

$$\omega_{p,q} = \sum_{i=0}^{p} \sum_{k=q+L-N}^{L-1} h_{i,k}. \quad \text{(D7)}$$

• Region 8 $(0 \le p \le M - 2, L - 1 \le q \le N - L)$: $\omega_{p,q}$'s which fall within this region are found in the 4th term of (A1), the 4th and 12th terms of (B1), the 4th term of (C3), and the 4th term of (C6). Adding these terms together we obtain

$$\sum_{p=0}^{M-2} \sum_{q=L-1}^{N-L} W^{pu+qv} \left\{ \sum_{i=1}^{M-p-1} \sum_{k=q-L+1}^{q} f_{-i,k} h_{p+i,q-k} - (1 - \omega_{p,q}) \sum_{i=0}^{p} \sum_{k=q-L+1}^{q} f_{i,k} h_{p-i,q-k} \right\}.$$

If $f_{i,k}$ varies slowly within regions $(-M + 1) \le i \le (M - 2)$ and $(-L + q + 1) \le k \le q$, the condition for the vanishing of the above expression is

$$\omega_{p,q} = \sum_{i=0}^{p} \sum_{k=0}^{L-1} h_{i,k}. \quad \text{(D8)}$$

Combining Eqs. (D1) to (D8) and the fact that $\omega_{p,q} = 1$ for $(M - 1) \le p \le (N - M)$ and $(L - 1) \le q \le (N - L)$, we have the window defined by (4) presented in Section III of this paper.

REFERENCES

1. W. K. Pratt, *Digital Image Processing*, Wiley, New York, 1978.
2. J. W. Woods, J. Biemond, and A. M. Tekalp, Boundary value problem in image restoration, in *Proceedings, ICASSP 85, Tampa, FL, 1985*, pp. 692–695.
3. H. Lim, K-C. Tan, and B. T. G. Tan, Edge errors in inverse and Wiener filter restorations of motion-blurred images and their windowing treatment, *Comput. Vision, Graphics, Image Process.* **53**, 1991, 186–195.
4. R. L. Lagendijk, J. Biemond, and D. E. Boekee, Regularized iterative image restoration with ringing reduction, *IEEE Trans. Acoust. Signal Speech Process.* **36**, 1988, 1874–1887.
5. M. I. Sezan and A. M. Tekalp, Iterative image restoration with ringing suppression using POCS, *Proceedings, IEEE Int. Conf. on ASSP, New York, 1988*, pp. 1874–1887.