PC1221 Fundamentals of Physics I

Lectures 5 and 6

Vectors

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Ground Rules

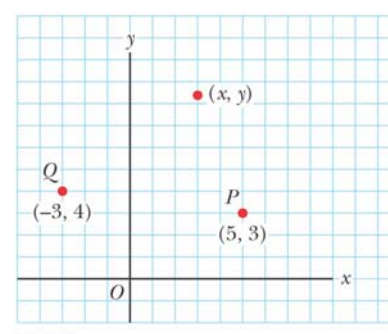
- Switch off your handphone and pager
- Switch off your laptop computer and keep it
- No talking while lecture is going on
- No gossiping while the lecture is going on
- Raise your hand if you have question to ask
- Be on time for lecture
- Be on time to come back from the recess break to continue the lecture
- Bring your lecturenotes to lecture

Coordinate Systems

- Used to describe the position of a point in space
- Coordinate system consists of
  - a fixed reference point called the origin, (0, 0) for 2-directional frame, and (0, 0, 0) for 3-directional frame
  - specific axes with scales and labels

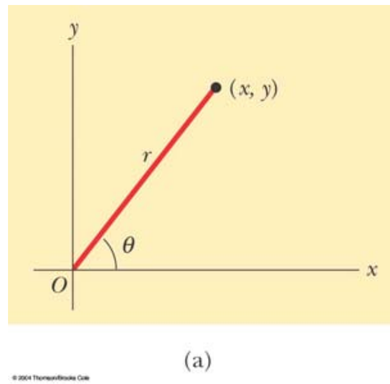
Cartesian Coordinate System

- Also called rectangular coordinate system
- *x*-axis and *y*-axis intersect at the origin
- Points are labeled (*x*,*y*)



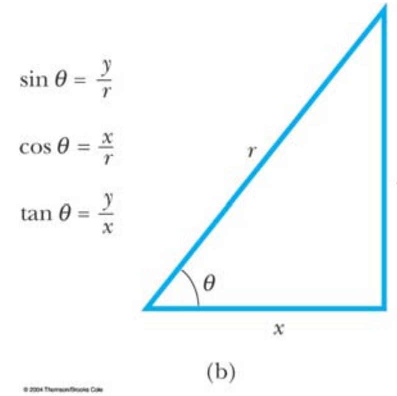
Polar Coordinate System

- Origin and reference line are noted
- Point is distance *r* from the origin in the direction of angle θ, counter-clock-wise (ccw) from reference line
- Points are labeled (r,θ)



Polar to Cartesian Coordinates

- Based on a right triangle formed by *r* and θ
- $x = r\cos\theta$
- $y = r\sin\theta$



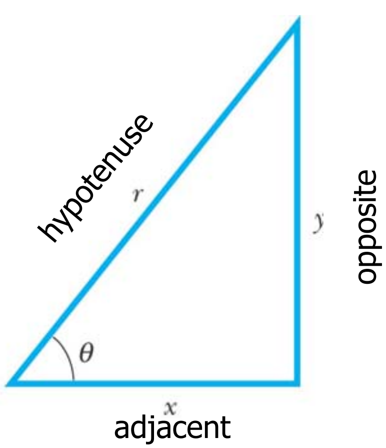
Cartesian to Polar Coordinates

- *r* is the hypotenuse and θ an angle

$$\tan\theta = \frac{y}{x}$$

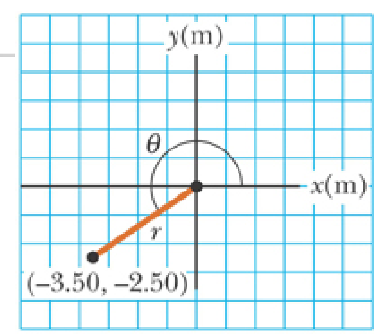
$$r = \sqrt{x^2 + y^2}$$

- θ must be ccw from positive *x* axis for these equations to be valid



Example

- The Cartesian coordinates of a point in the *xy* plane are $(x,y) = (-3.50, -2.50)$ m, as shown in the figure. Find the polar coordinates of this point.



Solution:

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3.50\text{ m})^2 + (-2.50\text{ m})^2} = 4.30\text{ m}$$

$$\tan\theta = \frac{y}{x} = \frac{-2.50\text{ m}}{-3.50\text{ m}} = 0.714$$

$$\theta = 216^\circ$$

Vectors and Scalars

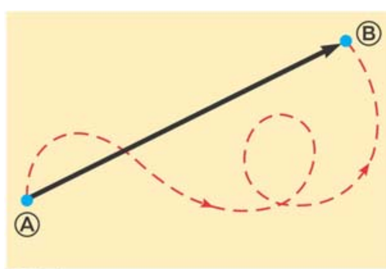
- A ***scalar quantity*** is completely specified by a single value with an appropriate unit and has no direction.
- A ***vector quantity*** is completely described by a number with appropriate units plus a direction.

Vector Notation

- When handwritten, use an arrow: $\vec{A}$
- When printed, will be in bold print: **A**
- When dealing with just the magnitude of a vector in print, an italic letter will be used: *A* or |**A**|
- The magnitude of the vector has physical units
- The magnitude of a vector is always a positive number

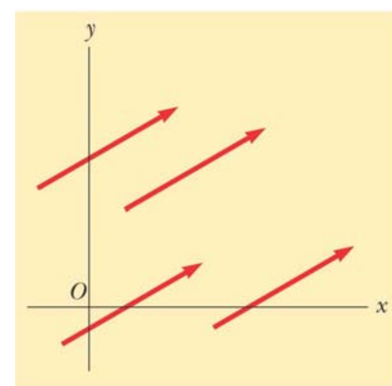
Vector Example

- A particle travels from A to B along the path shown by the dotted red line
  - This is the ***distance*** traveled and is a scalar
- The ***displacement*** is the solid line from A to B
  - The displacement is independent of the path taken between the two points
  - Displacement is a vector



Equality of Two Vectors

- Two vectors are ***equal*** if they have the same magnitude and the same direction
- $\mathbf{A} = \mathbf{B}$ if $A = B$ and they point along parallel lines
- All of the vectors shown are equal

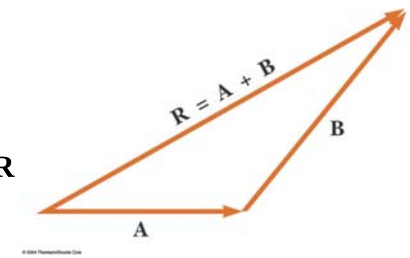


Adding Vectors

- When adding vectors, their directions must be taken into account
- Units must be the same
- Graphical Methods
  - Use scale drawings
- Algebraic Methods
  - More convenient

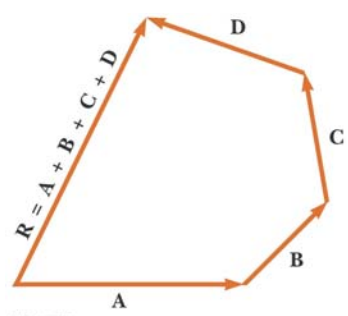
Adding Vectors Graphically

- Continue drawing the vectors "tip-to-tail"
- The resultant is drawn from the origin of **A** to the end of the last vector
- Measure the length of **R** and its angle
  - Use the scale factor to convert length to actual magnitude



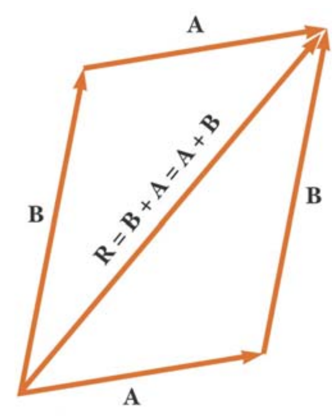
Adding Vectors Graphically, cont

- When you have many vectors, just keep repeating the process until all are included
- The resultant is still drawn from the origin of the first vector to the end of the last vector



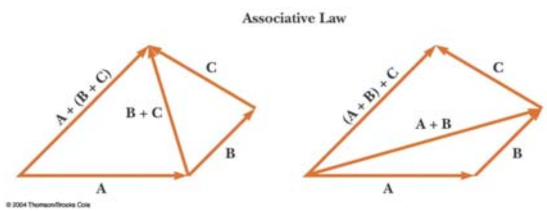
Adding Vectors, Rules

- When two vectors are added, the sum is independent of the order of the addition.
  - This is the ***commutative law of addition***
  - **A** + **B** = **B** + **A**



Adding Vectors, Rules cont.

- When adding three or more vectors, their sum is independent of the way in which the individual vectors are grouped
  - This is called the ***Associative Property of Addition***
  - **A** + (**B** + **C**) = (**A** + **B**) + **C**



Adding Vectors, Rules final

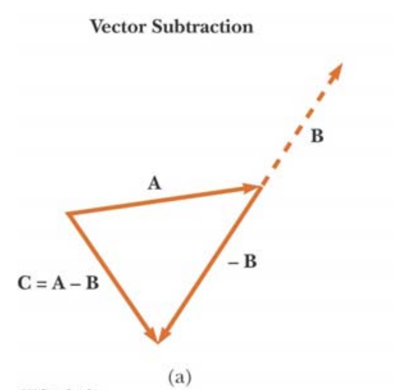
- When adding vectors, all of the vectors must have the same units
- All of the vectors must be of the same type of quantity
  - For example, you cannot add a displacement to a velocity

Negative of a Vector

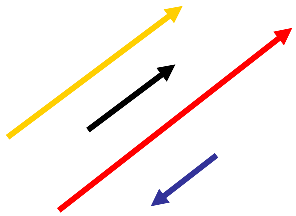
- The negative of a vector is defined as the vector that, when added to the original vector, gives a resultant of zero
  - Represented as –**A**
  - **A** + (-**A**) = 0
- The negative of the vector will have the same magnitude, but point in the opposite direction

Subtracting Vectors

- Special case of vector addition
- If **A** – **B**, then use **A**+ (-**B**)
- Continue with standard vector addition procedure



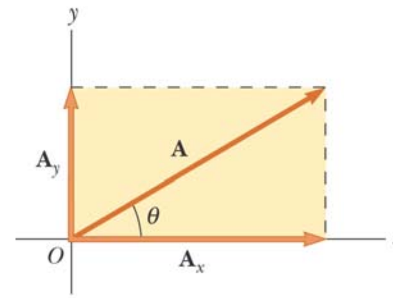
Multiplying or Dividing a Vector by a Scalar



- The result of the multiplication or division is a vector
- The magnitude of the vector is multiplied or divided by the scalar
- If the scalar is positive, the direction of the result is the same as of the original vector
- If the scalar is negative, the direction of the result is opposite that of the original vector

Components of a Vector

- A **component** is a part
- It is useful to use **rectangular components**
  - These are the projections of the vector along the x-axis and y-axis



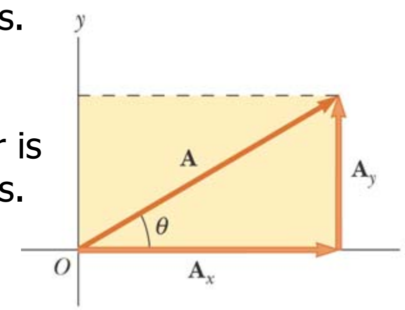
(a)

Vector Component Terminology

- $\mathbf{A_x}$ and $\mathbf{A_y}$ are the ***component vectors*** of $\mathbf{A}$
  - They are vectors and follow all the rules for vectors
- A_x and A_y are scalars, and their associated vectors $\mathbf{A_x}$ and $\mathbf{A_y}$ will be referred to as the components of $\mathbf{A}$

Components of a Vector, 2

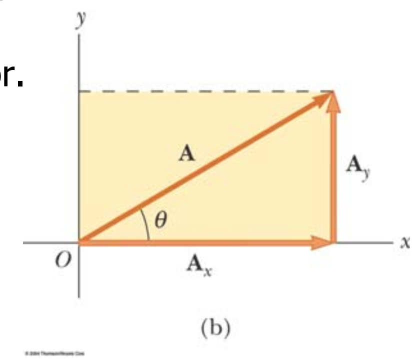
- The x-component of a vector is its projection along the x-axis. The magnitude is $A_x = A \cos \theta$
- The y-component of a vector is its projection along the y-axis. The magnitude is $A_y = A \sin \theta$
- Then, $\mathbf{A} = \mathbf{A_x} + \mathbf{A_y}$



(b)

Components of a Vector, 3

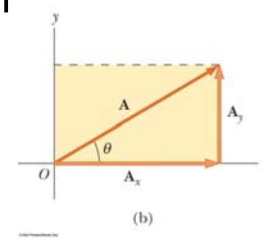
- The *y*-component vector is moved to the end of the *x*-component vector.
- This is due to the fact that any vector can be moved parallel to itself without being affected
  - This completes the triangle.



Components of a Vector, 4

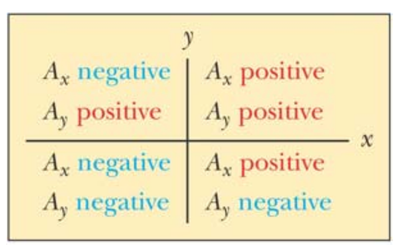
- The previous equations are valid ***only if θ is measured with respect to the x-axis***
- The components are the legs of the right triangle whose hypotenuse is **A**

$$A = \sqrt{A_x^2 + A_y^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{A_y}{A_x}$$



Components of a Vector, final

- The components can be positive or negative and will have the same units as the original vector
- The signs of the components will depend on the angle

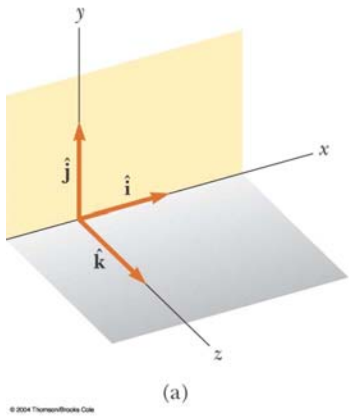


Unit Vectors

- A ***unit vector*** is a dimensionless vector with a magnitude of exactly 1.
- Unit vectors are used to specify a direction and have no other physical significance

Unit Vectors, cont.

- The symbols
  $\hat{\text{i}}, \hat{\text{j}}, \text{and } \hat{\text{k}}$
  represent unit vectors
- They form a set of mutually perpendicular vectors
- They follows right-hand rules



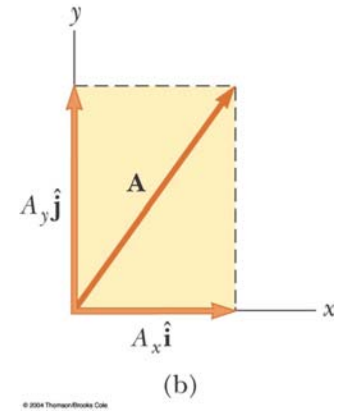
Unit Vectors in Vector Notation

- $\mathbf{A_x}$ is the same as $A_x\hat{\mathbf{i}}$ and $\mathbf{A_y}$ is the same as $A_y\hat{\mathbf{j}}$ etc.
- The complete vector can be expressed as

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$$

- If $\mathbf{A}$ is 3-directional, it is expressed as

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$



Adding Vectors Using Unit Vectors

- Using $\mathbf{R} = \mathbf{A} + \mathbf{B}$
- Then

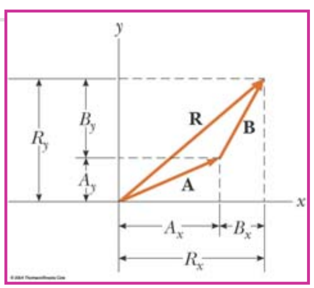
$$\mathbf{R} = \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}\right) + \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}\right)$$

$$\mathbf{R} = \left(A_x + B_x\right)\hat{\mathbf{i}} + \left(A_y + B_y\right)\hat{\mathbf{j}}$$

$$\mathbf{R} = \mathbf{R_x} + \mathbf{R_y}$$

- and so $R_x = A_x + B_x$ and $R_y = A_y + B_y$

$$R = \sqrt{R_x^2 + R_y^2} \qquad \theta = \tan^{-1}\frac{R_y}{R_x}$$



Adding Vectors Using Unit Vectors – Three Directions

- Using $\mathbf{R} = \mathbf{A} + \mathbf{B}$

$$\mathbf{R} = \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right) + \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}\right)$$

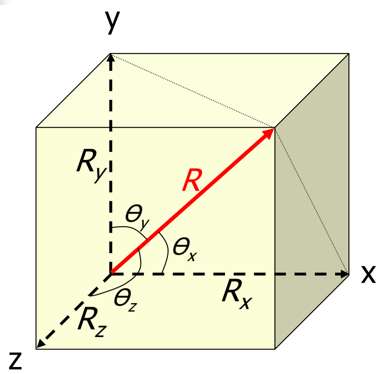
$$\mathbf{R} = \left(A_x + B_x\right)\hat{\mathbf{i}} + \left(A_y + B_y\right)\hat{\mathbf{j}} + \left(A_z + B_z\right)\hat{\mathbf{k}}$$

$$\mathbf{R} = \mathbf{R_x} + \mathbf{R_y} + \mathbf{R_z}$$

- $R_x = A_x + B_x$, $R_y = A_y + B_y$ and $R_z = A_z + B_z$

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} \qquad \theta_x = \text{ ?}$$

Angle of Vector in 3-D



$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

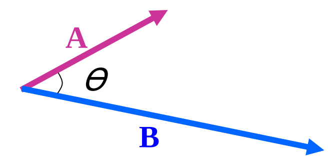
$$\theta_x = \cos^{-1} \frac{R_x}{R}$$

$$\theta_y = \cos^{-1} \frac{R_y}{R}$$

$$\theta_z = \cos^{-1} \frac{R_z}{R}$$

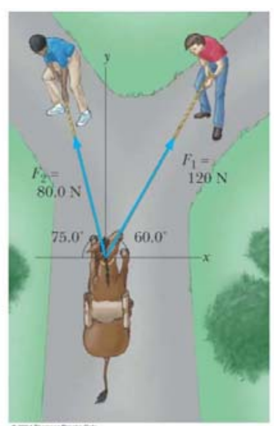
Dot Product of Two Vectors

$\mathbf{A} \bullet \mathbf{B} = |\mathbf{A}| \times |\mathbf{B}| \times \cos \theta$



Dot product of vectors is a scalar quantity.

Example. The helicopter view in Figure shows two men pulling on a stubborn donkey. Find (a) the single force that is equivalent to the two forces shown, and (b) the counter force that the donkey would have to apply in order to naturalize the two men. The forces are measured in units of newtons (abbreviated N).



Answer:

(a)

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$

$$\mathbf{F} = 120\cos(60.0°)\,\hat{\mathbf{i}} + 120\sin(60.0°)\,\hat{\mathbf{j}} - 80.0\cos(75.0°)\,\hat{\mathbf{i}} + 80.0\sin(75.0°)\,\hat{\mathbf{j}}$$

$$\mathbf{F} = 60.0\hat{\mathbf{i}} + 104\hat{\mathbf{j}} - 20.7\hat{\mathbf{i}} + 77.3\hat{\mathbf{j}} = \left(39.3\hat{\mathbf{i}} + 181\hat{\mathbf{j}}\right)\ \text{N}$$

$$|\mathbf{F}| = \sqrt{39.3^2 + 181^2} = \boxed{185\ \text{N}}$$

$$\theta = \tan^{-1}\left(\frac{181}{39.3}\right) = \boxed{77.8°}$$

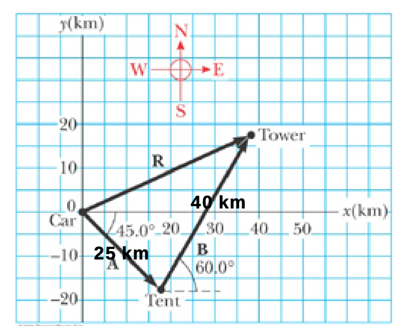
(b) Let F_3 be the counter force from the donkey

$$\mathbf{F}_3 = -\mathbf{F} = \boxed{\left(-39.3\hat{\mathbf{i}} - 181\hat{\mathbf{j}}\right)\ \text{N}}$$

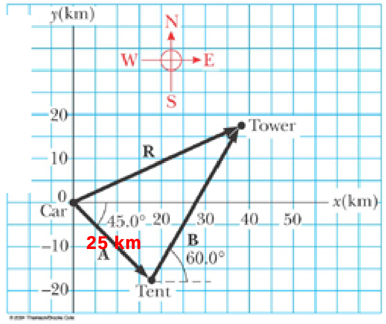
Example: A hiker begins a trip by first walking 25.0 km southeast from her car. She stops and sets up her tent for the night. On the second day, she walks 40.0 km in a direction 60.0° north of east, at which point she discovers a forest ranger's tower.

(A) Determine the components of the hiker's displacement for each day.

Solution: We can *categorize* this problem as an addition of two vectors.



Next, we analyze this problem by using our new knowledge of vector components. Displacement **A** has a magnitude of 25.0 km and is directed 45.0° below the positive *x* axis.



The components are:

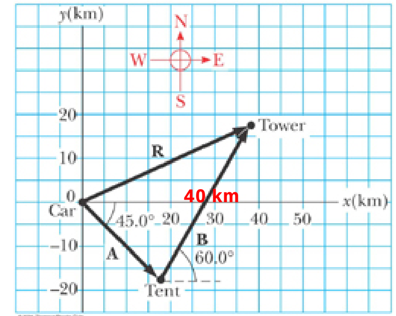
$$A_x = A\cos(-45.0°) = (25.0\text{ km})(0.707) = 17.7\text{ km}$$

$$A_y = A\sin(-45.0°) = (25.0\text{ km})(-0.707) = -17.7\text{ km}$$

$$\boxed{\begin{aligned}\cos(-\theta) &= \cos(\theta)\\ \sin(-\theta) &= -\sin(\theta)\end{aligned}}$$

The negative value of A_y indicates that the hiker walks in the negative *y* direction on the first day. The signs of A_x and A_y also are evident from the figure above.

The second displacement **B** has a magnitude of 40.0 km and is 60.0° north of east.



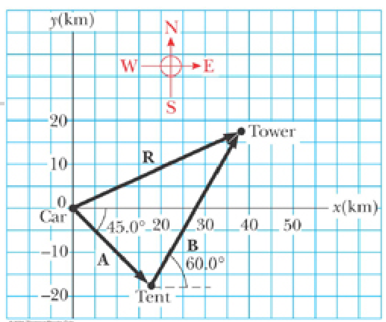
Its components are:

$$B_x = B\cos 60.0° = (40.0\text{ km})(0.500) = 20.0\text{ km}$$

$$B_y = B\sin 60.0° = (40.0\text{ km})(0.866) = 34.6\text{ km}$$

$$\boxed{\begin{aligned}&\sin(0°) = 0\\ &\sin(30°) = 0.5\\ &\sin(45°) = 0.7071 = \frac{1}{\sqrt{2}}\\ &\sin(60°) = 0.866 = \frac{\sqrt{3}}{2}\\ &\sin(90°) = 1\end{aligned}} \quad \boxed{\begin{aligned}&\cos(0°) = 1\\ &\cos(30°) = 0.866 = \frac{\sqrt{3}}{2}\\ &\cos(45°) = 0.7071 = \frac{1}{\sqrt{2}}\\ &\cos(60°) = 0.5\\ &\cos(90°) = 0\end{aligned}} \quad \boxed{\begin{aligned}&\tan(0°) = 0\\ &\tan(30°) = 0.5774 = \frac{1}{\sqrt{3}}\\ &\tan(45°) = 1\\ &\tan(60°) = 1.732 = \sqrt{3}\\ &\tan(90°) = \infty\end{aligned}}$$

(B) Determine the components of the hiker's resultant displacement **R** for the trip. Find an expression for **R** in terms of unit vectors.



Solution: The resultant displacement for the trip **R** = **A** + **B** has the following components:

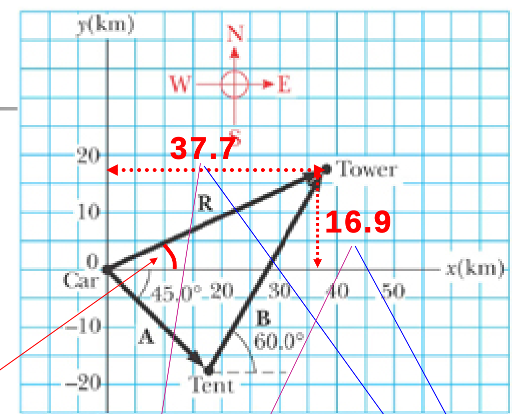
$$R_x = A_x + B_x = 17.7\text{ km} + 20.0\text{ km} = 37.7\text{ km}$$

$$R_y = A_y + B_y = -17.7\text{ km} + 34.6\text{ km} = 16.9\text{ km}$$

In unit-vector form, we can write the total displacement as

$$\mathbf{R} = (37.7\,\hat{\imath} + 16.9\,\hat{\jmath})\text{ km}$$

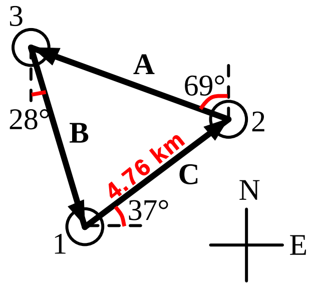
Using Equations for R and Θ, we find that <u>the vector **R** has a magnitude of 41.3 km and is directed 24.1° north of east</u>.



$$R = \sqrt{R_x^2 + R_y^2} \qquad \theta = \tan^{-1}\frac{R_y}{R_x}$$

Example. A ferry boat transports tourists among three islands. It sails from the first island to the second island, 4.76 km away, in a direction 37.0° north of east. It then sails from the second island to the third island in a direction 69.0° west of north. Finally it returns to the first island, sailing in a direction 28.0° east of south. Calculate the distance between (a) the second and third islands (b) the first and third islands.

Answer: One of the possible approaches is as follows:



Let A be the distance between islands (2) and (3).
Let B be the distance between islands (1) and (3).

For the x components, we have:

$4.76 \cos 37° - A \sin 69° + B \sin 28° = 0$

i.e., $3.80 - 0.934A + 0.469B = 0$

$0.469B = -3.80 + 0.934A$

➔ **$B = -8.10 + 1.99\,A$ --- (i)**

For the y components, we have:

$4.76 \sin 37° + A \cos 69° - B \cos 28° = 0$

i.e., **$2.86 + 0.358A - 0.883B = 0$ --- (ii)**

By substituting (i) into (ii), we have

$2.86 + 0.358\,A - 0.883\,(-8.10 + 1.99A) = 0$

$2.86 + 0.358A + 7.15 - 1.76A = 0$

$1.40A = 10$

$\underline{A = 7.14 \text{ km}}$

By substituting the value of A into (i), we have

$B = -8.10 + 1.99\,(7.14)$

$= \underline{6.11 \text{ km}}$