PC1221 Fundamentals of Physics I

Lectures 7 and 8
Motion in Two Dimensions

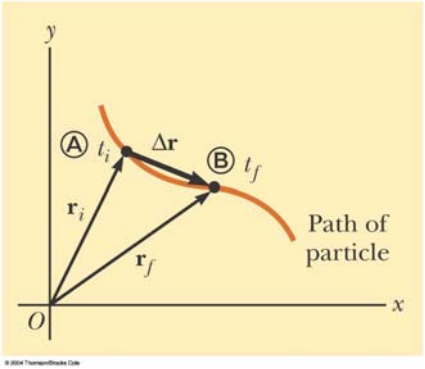
A/Prof Tay Seng Chuan

Ground Rules

- Switch off your handphone and pager
- Switch off your laptop computer and keep it
- No talking while lecture is going on
- No gossiping while the lecture is going on
- Raise your hand if you have question to ask
- Be on time for lecture
- Be on time to come back from the recess break to continue the lecture
- Bring your lecturenotes to lecture

Position and Displacement

- The position of an object is described by its position vector, **r**
- The **displacement** of the object is defined as the ***change in its position***
  - $\Delta\mathbf{r} = \mathbf{r}_f - \mathbf{r}_i$
- In two- or three-dimensional kinematics, everything is the same as in one-dimensional motion except that we must now use full vector notation



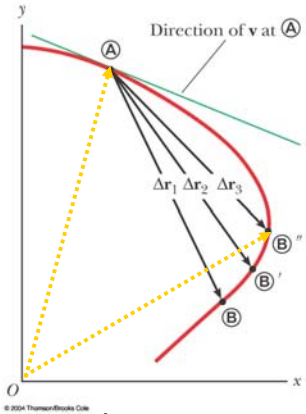
Plane View

Average Velocity

- The average velocity is the ratio of the displacement to the time interval for the displacement

$$\bar{\mathbf{v}} = \frac{\Delta\mathbf{r}}{\Delta t}$$

  - The direction of the average velocity is the direction of the displacement vector, $\Delta\mathbf{r}$
- The average velocity between points is *independent of the path* taken
  - It is dependent on the displacement

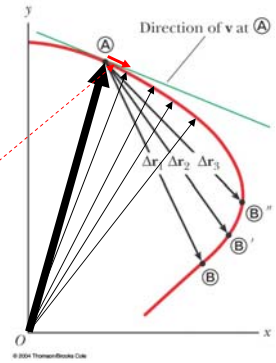


Plane View

Instantaneous Velocity

- The instantaneous velocity is the limit of the average velocity as Δt approaches zero

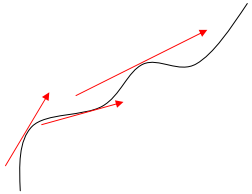
$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$



The direction of small displacement (change in positions) tells the direction that the particle is heading at the moment.

Instantaneous Velocity, cont

- The direction of the instantaneous velocity vector at any point in a particle's path is along a line tangent to the path at that point and in the direction of motion
- The magnitude of the instantaneous velocity vector is the speed
  - The speed is a scalar quantity



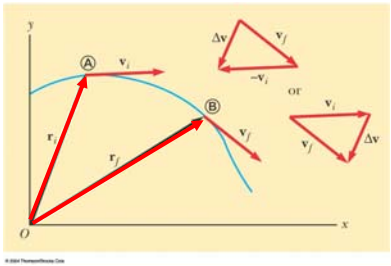
Average Acceleration

- The average acceleration of a particle as it moves is defined as the change in the instantaneous velocity vector divided by the time interval during which that change occurs.

$$\overline{\mathbf{a}} = \frac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i} = \frac{\Delta \mathbf{v}}{\Delta t}$$

Average Acceleration, cont

- As a particle moves, $\Delta \mathbf{v}$ can be found in different ways
- The average acceleration is a vector quantity directed along $\Delta \mathbf{v}$

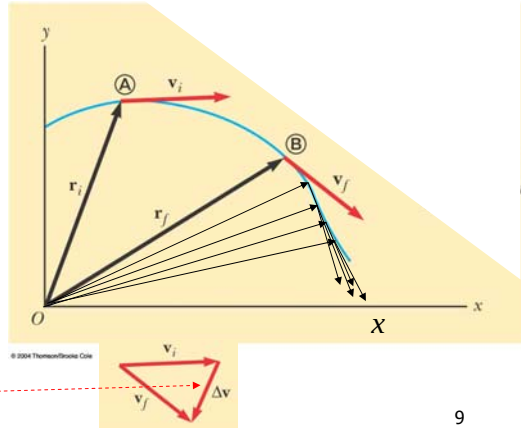


Plane View

Instantaneous Acceleration

- The instantaneous acceleration is the limit of the average acceleration as Δt approaches zero

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$$



The direction of small change in velocities tells the direction of acceleration.

Producing An Acceleration

- Various changes in a particle's motion may produce an acceleration, such as:
  - The magnitude of the velocity vector may change
  - The direction of the velocity vector may change
    - Even if the magnitude remains constant
  - Both may also change simultaneously

Kinematic Equations for Two-Dimensional Motion

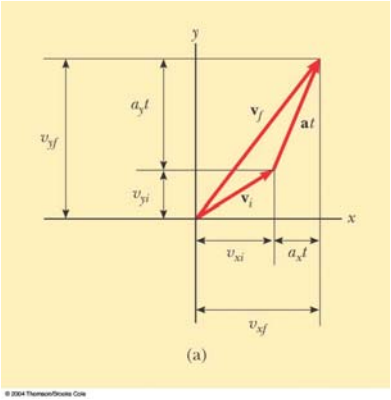
- When the two-dimensional motion has a constant acceleration, a series of equations can be developed that describe the motion
- These equations will be similar to those of one-dimensional kinematics

Kinematic Equations, 2

- Position vector $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$
- Velocity $\mathbf{v} = \frac{d\mathbf{r}}{dt} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$
  - Since acceleration is constant, we can also find an expression for the velocity as a function of time: $\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t$

Kinematic Equations, 3

- The velocity vector can be represented by its components
- $\mathbf{v}_f$ is generally not along the direction of either $\mathbf{v}_i$ or $\mathbf{a}t$



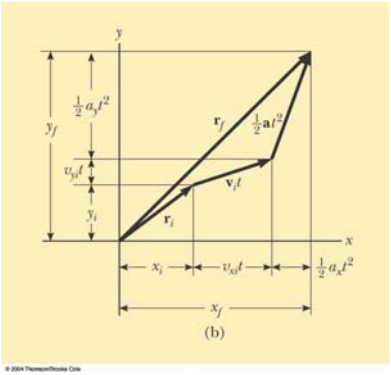
$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t$$

Kinematic Equations, 4

- The position vector can also be expressed as a function of time:
  - $\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \frac{1}{2}\mathbf{a}t^2$
  - This indicates that the position vector is the sum of three other vectors:
    - The initial position vector $\mathbf{r}_i$
    - The displacement resulting from $\mathbf{v}_i t$
    - The displacement resulting from $\frac{1}{2}\mathbf{a}t^2$

Kinematic Equations, 5

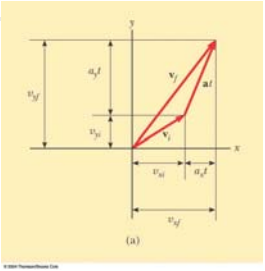
- The vector representation of the position vector
- $\mathbf{r}_f$ is generally not in the same direction as $\mathbf{v}_i$ or as $\mathbf{a}_i$
- $\mathbf{r}_f$ and $\mathbf{v}_f$ are generally not in the same direction

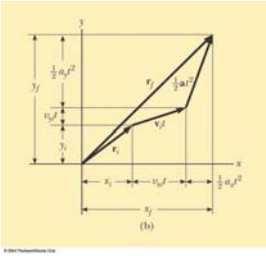


$$\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \frac{1}{2}\mathbf{a}t^2$$

Kinematic Equations, Components

- The equations for final velocity and final position are vector equations, therefore they may also be written in component form
- This shows that two-dimensional motion at constant acceleration is equivalent to two independent motions
  - One motion in the *x*-direction and the other in the *y*-direction



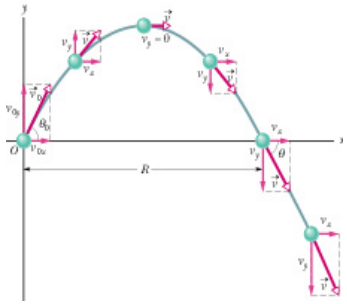


Kinematic Equations, Component Equations

- $\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t$ becomes
  - $v_{xf} = v_{xi} + a_x t$ and
  - $v_{yf} = v_{yi} + a_y t$
- $\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + ½ \mathbf{a}t^2$ becomes
  - $x_f = x_i + v_{xi} t + ½ a_x t^2$ and
  - $y_f = y_i + v_{yi} t + ½ a_y t^2$

Projectile Motion

- An object may move in both *x* and *y* directions simultaneously
- The form of two-dimensional motion we will deal with is called **projectile motion**



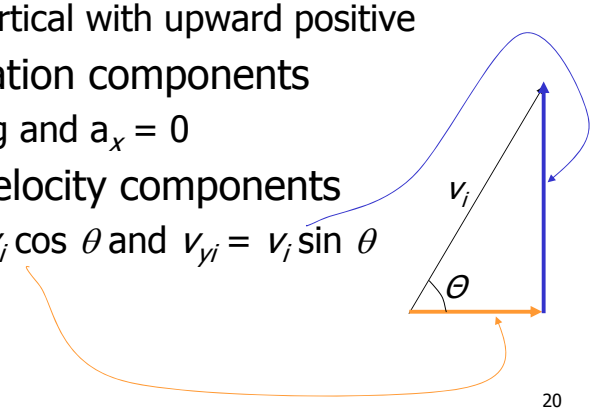
Front View

Assumptions of Projectile Motion

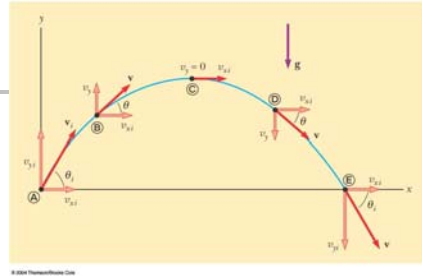
- The free-fall acceleration **g** is constant over the range of motion
  - **g** is directed downward
- The effect of air friction is negligible
- With these assumptions, an object in projectile motion will follow a parabolic path
  - This path is called the ***trajectory***

Verifying the Parabolic Trajectory

- Reference frame chosen
  - *y* is vertical with upward positive
- Acceleration components
  - $a_y = -g$ and $a_x = 0$
- Initial velocity components
  - $v_{xi} = v_i \cos\theta$ and $v_{yi} = v_i \sin\theta$



Verifying the Parabolic Trajectory, cont



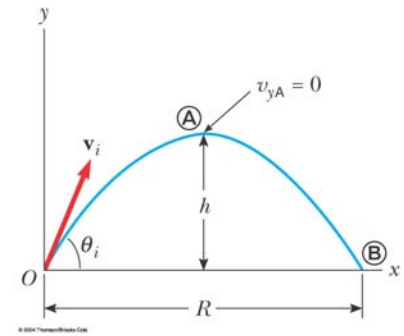
- Displacements
  - $x_f = v_{xi}\, t = (v_i \cos\theta)\, t$
  - $y_f = v_{yi}\, t + \frac{1}{2} a_y\, t^2 = (v_i \sin\theta) t - \frac{1}{2} g t^2$
- Combining the equations and removing t gives:

$$y = (\tan\theta_i)\, x - \left(\frac{g}{2 v_i^2 \cos^2\theta_i}\right) x^2$$

  - This is in the form of $y = ax - bx^2$ which is the standard form of a parabola

Range and Maximum Height of a Projectile

- When analyzing projectile motion, two characteristics are of special interest
- The range, R, is the horizontal distance of the projectile
- The maximum height the projectile reaches is h



Height of a Projectile, equation

- The maximum height of the projectile can be found in terms of the initial velocity vector:

$$h = \frac{v_i^2 \sin^2\theta_i}{2g}$$

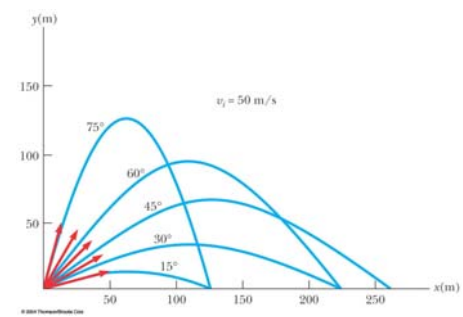
- This equation is valid only for symmetric motion

Range of a Projectile, equation

- The range of a projectile can be expressed in terms of the initial velocity vector:

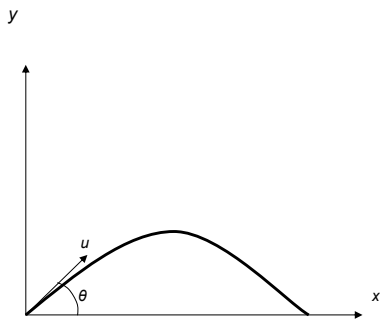
$$R = \frac{v_i^2 \sin 2\theta_i}{g}$$

- This is valid only for symmetric trajectory



We will derive them.

Projectile Formulation



Let the initial velocity be *u*. Let the angle subscripted by *u* and *x-axis* be *θ*.

Along the x-direction,

$$x = (u\cos\theta)\times t$$

$$\Rightarrow t = \frac{x}{u\cos\theta} \quad \text{— (1)}$$

Along the Y-direction,

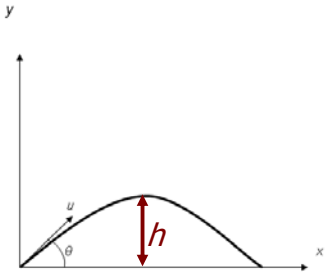
$$y = (u\sin\theta)\times t - \frac{1}{2}gt^2 \quad \text{— (2)}$$

Substitute (1) into (2):

$$y = (u\sin\theta)\times\frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x^2}{u^2\cos^2\theta}\right)$$

$$y = (\tan\theta)x - \frac{gx^2}{2u^2\cos^2\theta}$$

Maximum occurs when the y-component of the velocity is equal to 0, i.e. the object is at the moment of coming down.



$$\Rightarrow v = u\sin\theta - gt = 0$$

$$\therefore u\sin\theta - gt = 0$$

$$\Rightarrow t = \frac{u\sin\theta}{g} \quad \text{— (3)}$$

Substitute (3) into (2):

$$y = h = u\sin\theta\left(\frac{u\sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{u\sin\theta}{g}\right)^2$$

$$h = \frac{u^2\sin^2\theta}{g} - \frac{u^2\sin^2\theta}{2g}$$

$$\boxed{\Rightarrow h = \frac{u^2\sin^2\theta}{2g}} \text{ or } \boxed{\frac{v_i^2\sin^2\theta_i}{2g}}$$

To derive the maximum range (*R*) along the x-direction, we let $y = 0$ in equation 2.

$$\boxed{y = (u\sin\theta)\times t - \frac{1}{2}gt^2}$$

Why?

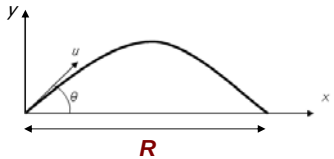
$$(u\sin\theta)\times t - \frac{1}{2}gt^2 = 0$$

$$\Rightarrow t\left(u\sin\theta - \frac{gt}{2}\right) = 0$$

$$\frac{gt}{2} = u\sin\theta$$

$$\therefore t = \frac{2u\sin\theta}{g} \quad \text{— (4)}$$

sin (A+B) = sin A cos B + cos A sin B



The distance travelled in x-direction for $t = \frac{2u\sin\theta}{g}$ seconds is :

$$x = (u\cos\theta)\times t$$

$$x = (u\cos\theta)\left(\frac{2u\sin\theta}{g}\right)$$

$$x = \frac{u^2}{g}\times(2\sin\theta\cos\theta)$$

$$\Rightarrow R = \frac{u^2\sin 2\theta}{g}$$

What is the maximum value of *R*, and the corresponding θ ?

Since $|\sin 2\theta| \le 1$, R is a maximum when

$\sin 2\theta = 1$, i.e., $2\theta = 90^\circ \Rightarrow \theta = 45^\circ$

When $\theta = 45^\circ$,

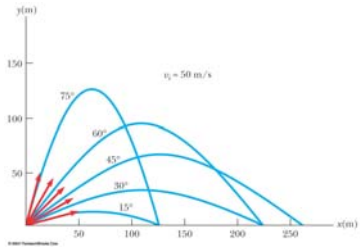
$$R = \frac{u^2}{g}\left(\sin\left(2\times 45^\circ\right)\right)$$

$$R = \frac{u^2}{g}\left(\sin 90^\circ\right)$$

$$\therefore R = \frac{u^2}{g}$$

Range of a Projectile, final

- The maximum range occurs at $\theta_i = 45^o$
- Complementary angles will produce the same range
  - But the maximum height will be different for the two angles
  - The times of the flight will be different for the two angles

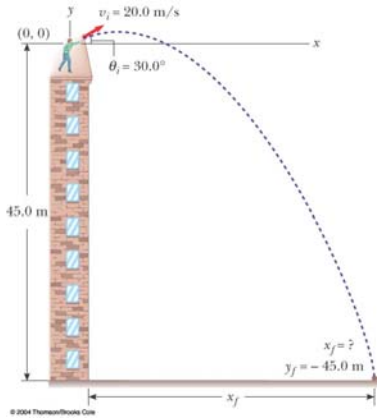


Projectile Motion – Problem Solving Hints

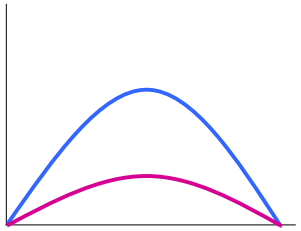
- Select a coordinate system
- Resolve the initial velocity into *x* and *y* components
- Analyze the horizontal motion using constant velocity techniques
- Analyze the vertical motion using constant acceleration techniques
- Remember that both directions share the same time

Non-Symmetric Projectile Motion

- Follow the general rules for projectile motion
- Break the *y*-direction into parts
  - up and down *or*
  - symmetrical back to initial height and then the rest of the height
- May be non-symmetric in other ways



Example. One strategy in a snowball fight is to throw a snowball at a high angle over level ground. Then, while your opponent is watching that snowball, you throw a second one at a low angle timed to arrive before or at the same time as the first one. Assume that both snowballs are thrown with a speed of 25.0 m/s. The first is thrown at an angle of 70.0° with respect to the horizontal. The release point of snowball and the hit point of the opponent are at the same level. (a) At what angle should the second snowball be thrown to arrive at the same point as the first? (b) How many seconds later should the second snowball be thrown after the first in order for both to arrive at the same time?



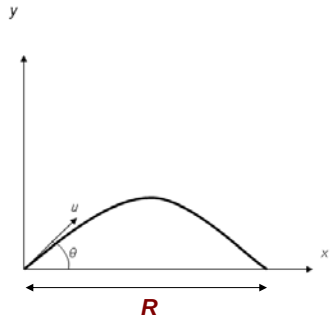
We have derived

$$R = \frac{u^2 \sin 2\theta}{g}$$

Since R, u, g are constant, $\sin 2\theta$ will also be a constant. **Let** the solution of 2θ be 2α , where $2\alpha \le 90$ °. We have

$$2\theta = 2\alpha, \text{ or } \quad 2\theta = 180° - 2\alpha \ .$$

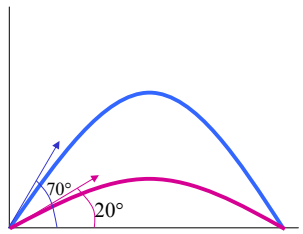
So, $\theta = \alpha$, or $\theta = 90° - \alpha$



Answer:

(a) The second snowball should be thrown at

$$90° - 70° = \boxed{20°}$$

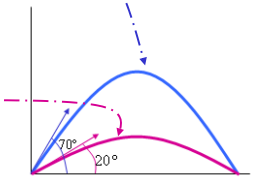


(b)

Applying $\Delta y = v_{0y}t + \frac{1}{2}a_y t^2$ to the vertical motion of the first snowball thrown at 70° gives $0 = \left[(25.0\ \text{m/s})\sin 70.0°\right]t_1 + \frac{1}{2}(-9.80\ \text{m/s}^2)t_1^2$.

Factoring this equation (remove the common term t_1), the non-zero solution of time of flight $t_1 = \frac{2(25.0\ \text{m/s})\sin 70.0°}{9.80\ \text{m/s}^2} = 4.79\ \text{s}$.

For the second snowball thrown at 20°, the time of flight is $t_2 = (5.10\ \text{s})\sin\theta = (5.10\ \text{s})\sin 20.0° = 1.74\ \text{s}$.

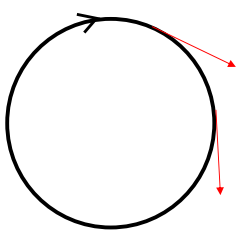


If they are to arrive simultaneously, the time delay between the first and second snowballs should be

$$\Delta t = t_1 - t_2 = 4.79\ \text{s} - 1.74\ \text{s} = \boxed{3.05\ \text{s}}$$

Uniform Circular Motion

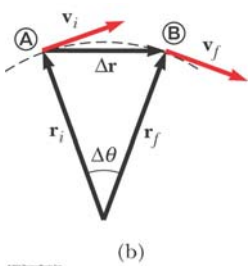
- ***Uniform circular motion*** occurs when an object moves in a circular path with a constant speed
- An acceleration exists since the *direction* of the motion is changing
  - This change in velocity is related to an acceleration
- The velocity vector is always tangent to the path of the object



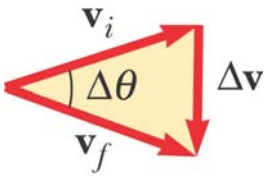
Plane View

Changing Velocity in Uniform Circular Motion

- The change in the velocity vector is due to the change in direction

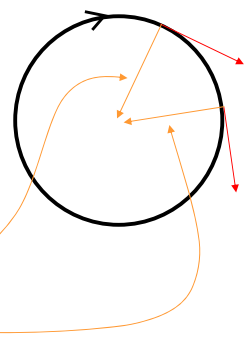


- The vector diagram shows $\Delta\mathbf{v} = \mathbf{v}_f - \mathbf{v}_i$



Centripetal Acceleration

- The acceleration is always perpendicular to the path of the motion
- The acceleration always points toward the center of the circle of motion
- This acceleration is called the ***centripetal acceleration***



Centripetal Acceleration, cont

- The magnitude of the centripetal acceleration vector is given by

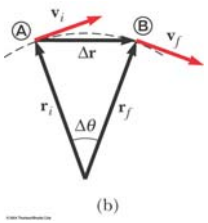
$$a_C = \frac{v^2}{r}$$

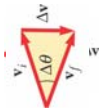
- By the use of same ratio, this can be derived as follows:

$$\frac{\Delta v}{v} = \frac{\Delta r}{r}$$

$$\frac{\Delta v}{\Delta r} = \frac{v}{r}$$

$$\frac{\Delta v}{\Delta t} = \frac{v}{r} \times \frac{\Delta r}{\Delta t} \qquad a_C = \frac{v^2}{r}$$





Period

- The ***period***, *T*, is the time required for one complete revolution (one complete cycle)
- As one complete cycle is the distance of the circumference. The speed of the particle would be the circumference of the circle of motion divided by the period. Speed = distance/time, so time = distance/speed
- Therefore, the period is $T = \frac{2\pi r}{v}$

Example. The astronaut orbiting the Earth in Figure is preparing to dock with a Westar VI satellite. The satellite is in a circular orbit 600 km above the Earth's surface, where the free-fall acceleration is 8.21 m/s^2. Take the radius of the Earth as 6400 km. Determine the speed of the satellite and the time interval required to complete one orbit around the Earth.

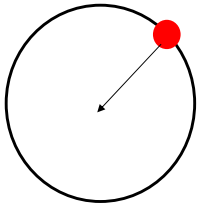
The satellite is in free fall. Its acceleration is due to gravity and is by effect a centripetal acceleration.

$$a_c = g$$

8.21 m/s²

so

$$\frac{v^2}{r} = g.$$



Solving for the velocity, $v = \sqrt{rg} = \sqrt{(6{,}400 + 600)(10^3\ \text{m})(8.21\ \text{m/s}^2)} = \boxed{7.58 \times 10^3\ \text{m/s}}$

Speed of satellite in order to maintain the orbit

$$v = \frac{2\pi r}{T}$$

and

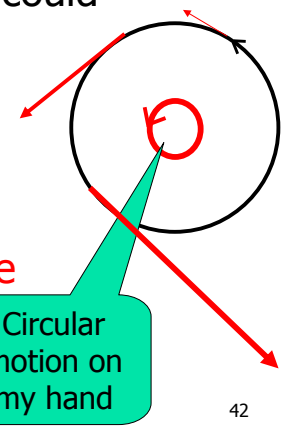
$$T = \frac{2\pi r}{v} = \frac{2\pi(7{,}000 \times 10^3\ \text{m})}{7.58 \times 10^3\ \text{m/s}} = \boxed{5.80 \times 10^3\ \text{s}}$$

$$T = 5.80 \times 10^3\ \text{s}\left(\frac{1\ \text{min}}{60\ \text{s}}\right) = 96.7\ \text{min}.$$

Time to complete 1 cycle of orbit

Tangential Acceleration

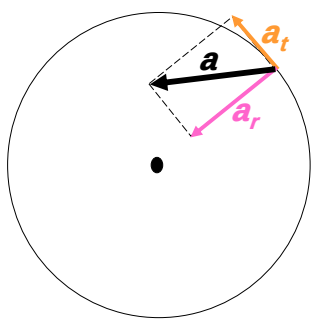
- The magnitude of the velocity could also be changing
- In this case, there would be a ***tangential acceleration***
- Observe what happens at the center (my hand) when I make the ball to rotate faster



Total Acceleration $a = a_t + a_r$

- The tangential acceleration causes the change in the speed of the particle
- The radial acceleration comes from a change in the direction of the velocity vector.

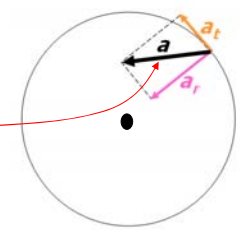
(In the textbook by Serway, the direction or radial acceleration is away from center – not important.)



Total Acceleration, equations

- The tangential acceleration: $a_t = \frac{dv}{dt}$
- The radial acceleration: $a_r = a_c = \frac{v^2}{r}$
- The total acceleration:
  - Magnitude

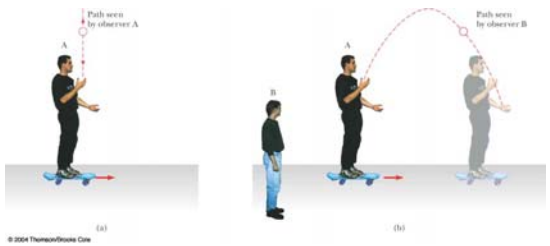
$$a = \sqrt{a_r^2 + a_t^2}$$



Why the string does not slack and the ball does not fly towards the center?

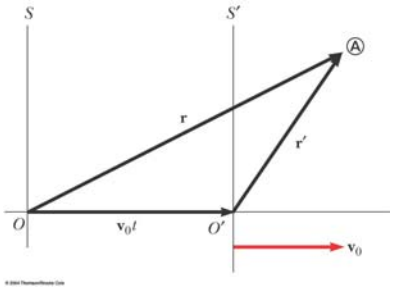
Relative Velocity

- Two observers moving relative to each other generally do not agree on the outcome of an experiment
- For example, observers A and B below see different paths for the ball



Relative Velocity, generalized

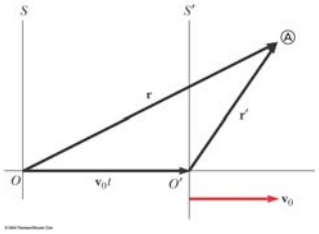
- Reference frame S is stationary
- Reference frame S' is moving at $\mathbf{v_0}$
  - This also means that S moves at $-\mathbf{v_0}$ relative to S'
- Define time $t = 0$ as that time when the origins coincide



Relative Velocity, equations

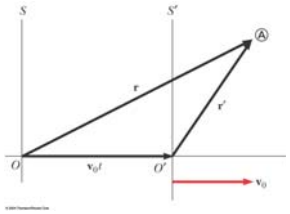
- The positions as seen from the two reference frames ($\mathbf{r}$ is the position seen from stationary frame, $\mathbf{r}'$ seen from moving frame) are related through the velocity
  - $\mathbf{r}' = \mathbf{r} - \mathbf{v_0}\, t$
- The derivative of the position equation will give the velocity equation
  - $\mathbf{v}' = \mathbf{v} - \mathbf{v_0}$
- These are called the **Galilean transformation equations**

(d1-9)



Acceleration in Different Frames of Reference

- The derivative of the velocity equation will give the acceleration equation
- The acceleration of the particle measured by an observer in one frame of reference is the same as that measured by any other observer moving at a *constant velocity* relative to the first frame. Why?

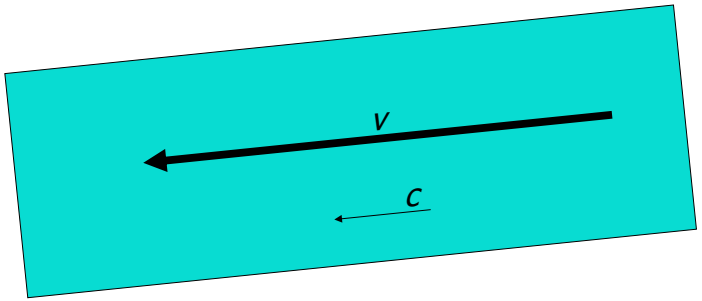


$$\mathbf{r}' = \mathbf{r} - \mathbf{v_0}\, t$$
$$\mathbf{v}' = \mathbf{v} - \mathbf{v_0}$$

$\mathbf{v_0}$ is a constant, so $\mathbf{v_0}$ has no effect on the change in velocity, ie, $\mathbf{a}' = \mathbf{a}$.

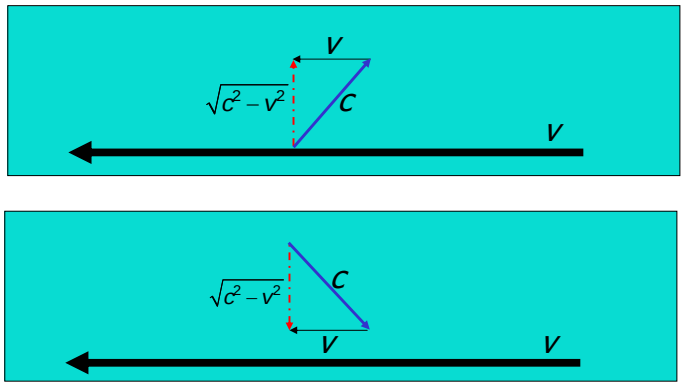
Example. Two swimmers, Tom and Jerry, start together at the same point on the bank of a wide stream that flows with a speed v. Both move at the same speed c ($c > v$), relative to the water. Tom swims downstream a distance L and then upstream the same distance. Jerry swims so that his motion relative to the Earth is perpendicular to the banks of the stream. He swims the distance L and then back the same distance, so that both swimmers return to the starting point. Which swimmer returns first?

Answer:



For Tom, his speed downstream is $c + v$, while his speed upstream is $c - v$.

Therefore, the total time for Tom is $t_1 = \frac{L}{c+v} + \frac{L}{c-v} = \boxed{\frac{\frac{2L}{c}}{1-\frac{v^2}{c^2}}}$



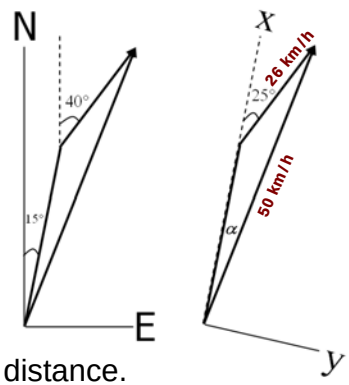
For Jerry, his cross-stream speed (both ways) is $\sqrt{c^2 - v^2}$

Thus, the total time for Jerry is $t_2 = \frac{2L}{\sqrt{c^2-v^2}} = \boxed{\frac{\frac{2L}{c}}{\sqrt{1-\frac{v^2}{c^2}}}}$

$t_1 = \frac{L}{c+v} + \frac{L}{c-v} = \boxed{\frac{\frac{2L}{c}}{1-\frac{v^2}{c^2}}}$

Since $1-\frac{v^2}{c^2} < 1$, $t_1 > t_2$, or Jerry, who swims cross-stream, returns first.

Example. A Coast Guard cutter detects an unidentified ship at a distance of 20.0 km in the direction 15.0° east of north. The ship is traveling at 26.0 km/h on a course at 40.0° east of north. The Coast Guard wishes to send a speedboat to intercept the vessel and investigate it. If the speedboat travels 50.0 km/h, in what direction should it head? Express the direction as a compass bearing with respect to due north.



Answer:

Choose the x-axis along the 20-km distance. The y-components of the displacements of the ship and the speedboat must agree:

$$(26\ \text{km/h})\,t\sin(40° - 15°) = (50\ \text{km/h})\,t\sin\alpha$$

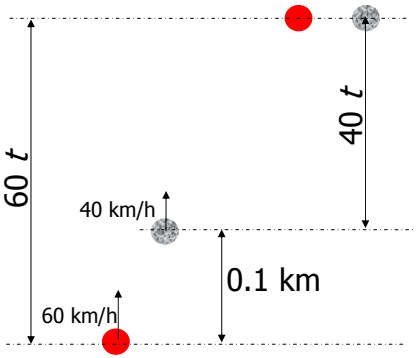
$$\alpha = \sin^{-1}\frac{11.0}{50} = 12.7°\,.$$

The speedboat should head

$$15° + 12.7° = \boxed{27.7°\ \text{east of north}}$$

Example. How long does it take an automobile traveling in the left lane at 60.0 km/h to pull alongside a car traveling in the right lane at 40.0 km/h if the cars' front bumpers are initially 100 m apart? (Disk 2)

Answer: The bumpers are initially 100 m apart (0.1 km). After time t the bumper of the leading car travels $40.0t$ km, while the bumper of the chasing car travels $60.0t$ km. Since the cars are side by side at time t we have



$$0.1 + 40t = 60t$$

$$t = 0.1/20 = 0.005\ hr$$
$$= 0.005 \times 3600\ sec$$
$$= 18\ s$$

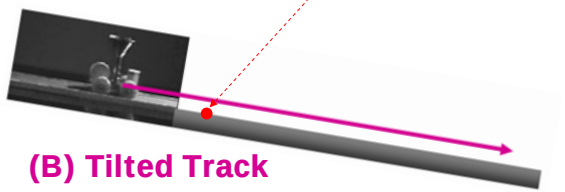
Examples.

In this demonstration a small car will roll down a track at constant speed. When the car reaches this point, it will fire a ball straight up as it continues to move. After the ball is fired, will it come down ahead of, on top of, or behind the car?

(A) Horizontal Track

A small car with a ball-firing mechanism on the top runs down a tilted track so that it is constantly accelerating. When it reaches this point it will fire the ball straight out of the cannon and continue on.

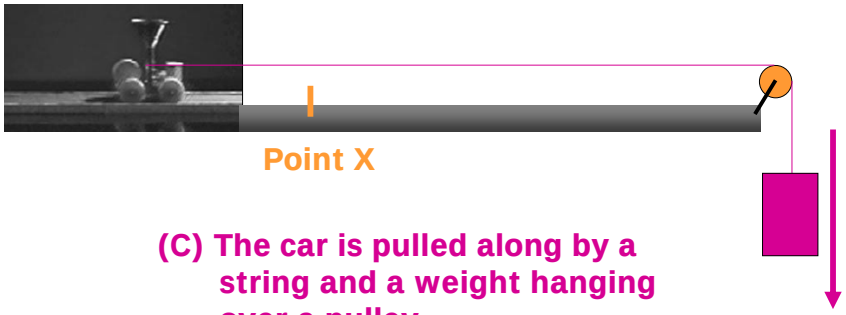
After the ball is fired, will it land ahead of, on top of, or behind the car?



(B) Tilted Track

Examples.

This time the car will be pulled along the horizontal track by a string and a weight hanging over a pulley, and will be accelerating constantly. The ball is fired at point X. Where will the ball land this time? (The horizontal track is long enough for the car not to fall off when the ball lands.)



(C) The car is pulled along by a string and a weight hanging over a pulley.