PC1221 Fundamentals of Physics I

Lectures 17 and 18

Linear Momentum and Collisions

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Ground Rules

- Switch off your handphone and pager
- Switch off your laptop computer and keep it
- No talking while lecture is going on
- No gossiping while the lecture is going on
- Raise your hand if you have question to ask
- Be on time for lecture
- Be on time to come back from the recess break to continue the lecture
- Bring your lecturenotes to lecture

Linear Momentum

- The **linear momentum** of a particle or an object that can be modeled as a particle of mass m moving with a velocity **v** is defined to be the product of the mass and velocity:
  - $\mathbf{p} = m\,\mathbf{v}$
    - The terms momentum and linear momentum will be used interchangeably in this course, i.e., when we say momentum we also means linear momentum (which is in a straight line)

Linear Momentum, cont

- Linear momentum is a vector quantity
  - Its direction is the same as the direction of **v**
- The dimensions of momentum (mass x velocity) are ML/T
- The SI units of momentum are kg · m / s
- Momentum can be expressed in component form (small letter p):
  - $p_x = m\,v_x$ $\quad p_y = m\,v_y$ $\quad p_z = m\,v_z$

Momentum in x direction

Momentum in y direction

Momentum in z direction

Newton and Momentum

- Newton called the product *mv* the *quantity of motion* of the particle
- Newton's Second Law can be used to relate the momentum of a particle to the resultant force acting on it

$$\Sigma \mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt}$$

with constant mass

$$\boxed{\Sigma \mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt}}$$

- The time rate of change of the linear momentum of a particle is equal to the net force acting on the particle
  - This is the form in which Newton presented the Second Law
  - It is a more general form than the one we used previously
  - This form also allows for mass changes
- Momentum approach can be used to analyse the motion in a system of particles

Conservation of Linear Momentum

- Whenever two or more particles in an isolated system interact, the total momentum of the system remains constant
  - The momentum of the *system* is conserved, but the momentum of individual particle may not necessarily conserved.
  - The **total** momentum of an isolated system equals its **initial** momentum

Conservation of Momentum, 2

- Conservation of momentum can be expressed mathematically in various ways
  - $\mathbf{p}_{total} = \mathbf{p}_1 + \mathbf{p}_2 =$ constant
  - $\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f}$

Initial Sum

final Sum

- In component form for the various directions, the total momentum in each direction is independently conserved
  - $p_{ix} = p_{fx}$ $\quad p_{iy} = p_{fy}$ $\quad p_{iz} = p_{fz}$
- Conservation of momentum can be applied to systems with any number of particles

Conservation of Momentum, Archer Example

- The archer is standing on a frictionless surface (ice). We know the mass of the archer (with bow) and the mass of the arrow, and the speed of the arrow. What will be the recoil speed of the archer?
- Approaches to solve this problem:
  - Newton's Second Law – no, no information about *F* or *a*
  - Energy approach – no, no information about work or energy
  - Momentum – yes

- Let the system be the archer with bow (particle 1) and the arrow (particle 2)
- There are no external forces in the *x*-direction, so it is isolated in terms of momentum in the *x*-direction
- Total momentum before releasing the arrow is 0
- The total momentum after releasing the arrow is

$$p_{1f} + p_{2f} = 0$$

$$p_{1f} + p_{2f} = 0, \text{ or,}$$

$$m_1 v_{1f} + m_2 v_{2f} = 0$$

- The archer will move in the opposite direction of the arrow after the release
  - Agrees with Newton's Third Law
- Because the archer is much more massive than the arrow, his acceleration and velocity will be much smaller than those of the arrow

Impulse and Momentum

- From Newton's Second Law

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt} = \frac{F}{1}$$

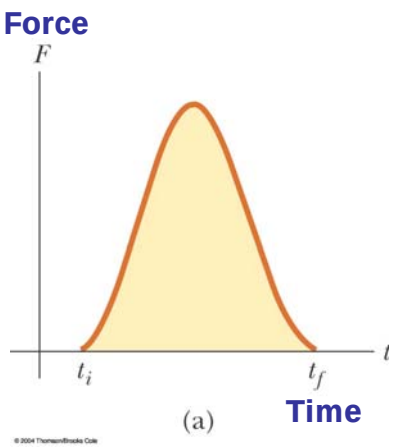
- Solving for $d\mathbf{p}$ (by cross multiplying) gives $d\mathbf{p} = \mathbf{F}dt$
- By integration, we can find the change in momentum over some time interval

$$\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f} \mathbf{F}dt = \mathbf{I}$$

- The integral is called the ***impulse*** (***I***)of the force **F** acting on an object over the time Δt
- The impulse imparted to a particle by a force is equal to the change in the momentum of the particle (**impulse-momentum theorem**). This is equivalent to Newton's Second Law.

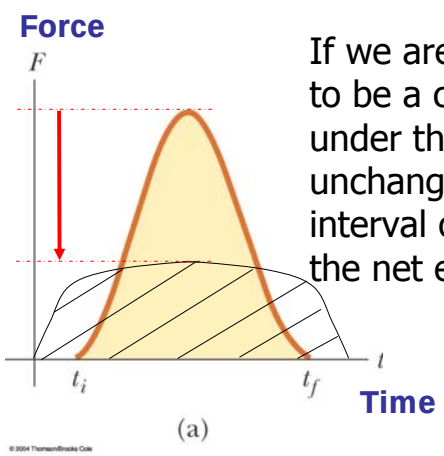
$$\Delta \mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f} \mathbf{F} dt = \mathbf{I}$$

- Impulse is a vector quantity
- The magnitude of the impulse is equal to the area under the force-time curve
- Dimensions of impulse are
  M (L T^{-2}) T = M L T^{-1}
  = M L / T
  ([] signs removed for simplicity)
- Impulse is not a property of the particle, but a measure of the change in momentum of the particle



If I throw an egg at you, how are you going to catch it without messing out yourself?

$$\Delta \mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f} \mathbf{F} dt = \mathbf{I}$$

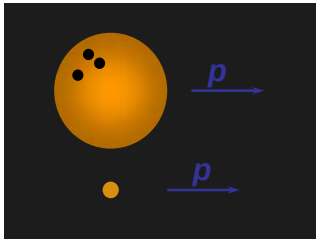


If we are to keep the impulse to be a constant, i.e., the area under the curve is to remain unchanged, but we extend the interval of (t_f- t_i), what will be the net effect?

Example. A bowling ball and a Ping-Pong ball are rolling toward you with the same momentum. If you exert the same force to stop each one, which one will take a longer time to bring it to rest?

1) the bowling ball
2) same time for both
3) the Ping-Pong ball
4) impossible to say

Answer:



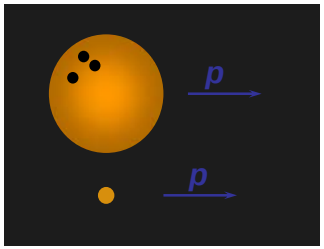
We know: $\Delta p = F_{av} \Delta t$, so $\Delta t = \dfrac{\Delta p}{F_{av}}$.

Here, F and Δp are the *same* for both balls! It will take the *same amount of time* to stop them.

Example. A bowling ball and a Ping-Pong ball are rolling toward you with the same momentum. If you exert the same force to stop each one, which *stopping distance* is longer?

1) the bowling ball
2) same distance for both
3) the Ping-Pong ball
4) impossible to say

Answer:



Both rolling balls have velocity thus they have kinetic energy of $\frac{1}{2}mv^2$. Since the momentum is the same for both balls, the ball with lesser mass has the larger speed (why? mv!), and thus the larger KE ($\frac{1}{2}mv^2 = \frac{1}{2}(mv)v$). In order to remove that KE, work must be done, where W = *Fd* = $\frac{1}{2}mv^2$. Because the force is the *same* in both cases, the distance needed to stop the less massive ball (larger KE) must be *longer*.

Example. A garden hose is held as shown. The hose is originally full of motionless water. What additional force is necessary to hold the nozzle stationary after the water flow is turned on, if the discharge rate is 0.600 kg/s with a speed of 25.0 m/s?

Answer:

The force exerted by water on the nozzle (thus the man) is

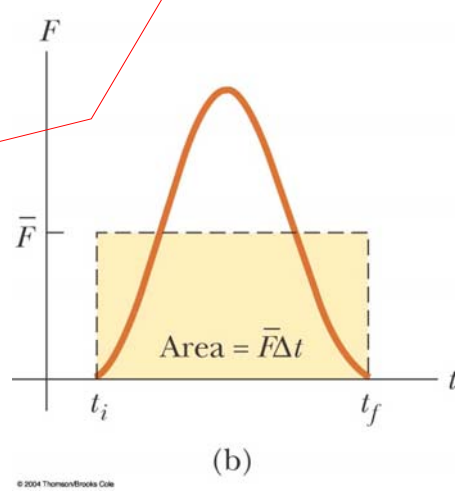
$$\Delta p = Fx\Delta t \Rightarrow F = \frac{\Delta p}{\Delta t} = \frac{mv_f - mv_i}{\Delta t} = \frac{m(v_f - v_i)}{\Delta t}$$

$$F = \frac{m}{\Delta t}x(v_f - v_i) = 0.6x(25-0) = 15N$$ (to the right in this diagram)

The nozzle will accelerate to the right. Thus, the gardener must apply a 15.0 N force (to the left) to hold the hose stationary.

$$\Delta \mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f} \mathbf{F}dt = \mathbf{I} = \bar{\mathrm{F}}\,\Delta\mathbf{t}$$

- The impulse can also be found by using the time and averaged force
- $\mathbf{I} = \bar{\mathrm{F}}\Delta t$
- This would give the same impulse as the time-varying force does

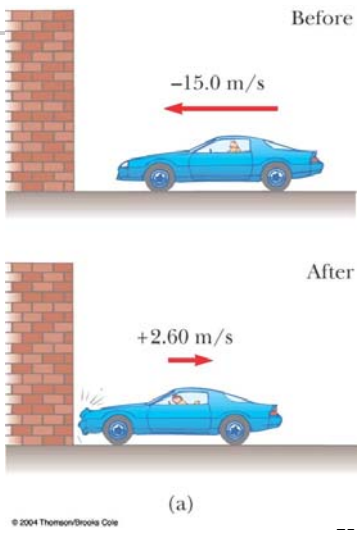


Impulse Approximation

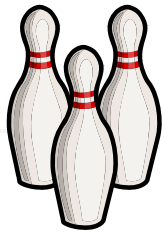
- In many cases of collision, one force acting on a particle will be much greater than any other force acting on the particle. When using the Impulse Approximation, we will assume this is true. The force will be called the impulse force. This approximation is very useful when the duration of collision is very short as we can ignore all other forces. E.g., when a baseball is struck by a bat, the time of the collision is about 0.01 second, and the force exerted by the bat on the ball is several thousand newtons so we can ignore the effect of their weights. This will simply our model. $\mathbf{p}_i$ and $\mathbf{p}_f$ represent the momenta (momentums) immediately before and after the collision respectively
- The particle is assumed to move very little during the collision (at the short time interval of contact)

Impulse-Momentum: Crash Test Example

- The momenta before and after the collision between the car and the wall can be determined ($\mathbf{p} = m\,\mathbf{v}$)
- Find the impulse:
  - $\mathbf{I} = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i$
  - $\mathbf{F} = \Delta\mathbf{p} / \Delta t$



The Fun Fair



You have paid $10 to a stall owner at a fun fair. You have to knock down 3 pins arranged in a triangle from a distance. If you can do it you will be given a teddy bear. The stall owner gives you two items before you proceed:

(i) rounded bean bag

(ii) rubber ball

Both of them have same mass and same radius. You are allowed to use only one of the items to knock the pins down. Which item should you use? Why?

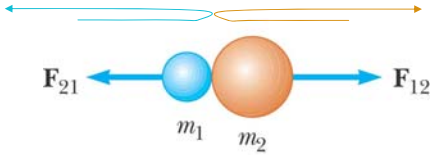
Answer:

The more effective item to be used to knock down the pins is the rubber ball. The bouncy rubber ball will bounce back and continue to exert a force on the pin. The impulse (force x time) delivered by the rubber ball will be greater than delivered by the bean bag because the rubber ball will exert its forward force for a longer time (during contact and rebounding). The ball will rebound with its momentum reversed, having transferred roughly twice its original momentum to the pins.

Collisions – Characteristics

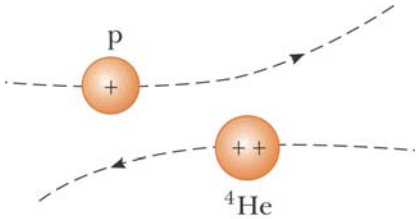
- We use the term **collision** to represent an event during which two particles come close to each other and interact by means of forces
- The time interval during which the velocity changes from its initial to final values is assumed to be very short

Collisions – Example 1



- Collisions may be the result of direct contact
- Its momentum is conserved

Collisions – Example 2



- Another type of collision needs not include physical contact between the objects. E.g., the collision of a proton, and a Alpha particle (the nucleus of a helium atom). As both are positively charged, they repel each other due to the strong electrostatic force between them at close separation and never come into physical contact.
- There are still forces between the particles
- This type of collision can be analyzed in the same way as those that include physical contact

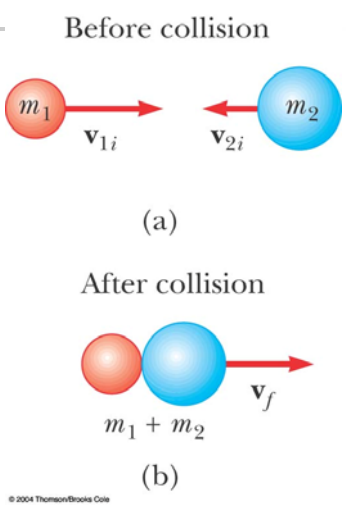
Types of Collisions

- In an ***inelastic*** collision, kinetic energy is not conserved although momentum is still conserved
  - If the objects stick together after the collision, it is a ***perfectly inelastic*** collision
- In an ***elastic*** collision, momentum and kinetic energy are conserved
  - Perfectly elastic collisions occur on a microscopic level (eg, atomic or sub-atomic level)
  - In macroscopic (eg, what you can see with your eyes) collisions (car accident, your friends knock on you, etc.), only approximately elastic collisions actually occur. E.g., some energy is lost in heat (sparks) or sound during collision

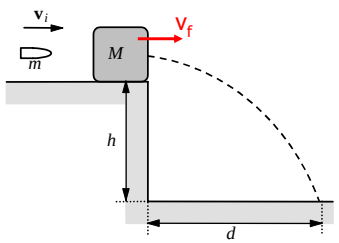
Perfectly Inelastic Collisions

- Since the objects stick together, they share the same velocity after the collision
- $m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = (m_1 + m_2)\,\mathbf{v}_f$

$$\mathbf{v}_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}$$



Example. A bullet of mass m is fired into a block of mass M initially at rest at the edge of a frictionless table of height h. The bullet remains in the block, and after impact the block lands a distance d from the bottom of the table. Determine the initial speed of the bullet.



Answer: Using conservation of momentum from just before to just after the impact of the bullet with the block:

$$m v_i = (M + m)\, v_f \quad \text{or} \quad v_i = \left(\frac{M+m}{m}\right) v_f \qquad (1)$$

The speed of the block and embedded bullet just after impact may be found using kinematic equations: $d = v_f t$ and $h = \frac{1}{2} g t^2$

Thus, $t = \sqrt{\frac{2h}{g}}$ and $v_f = \frac{d}{t} = d\sqrt{\frac{g}{2h}} = \sqrt{\frac{gd^2}{2h}}$

Substituting into (1) from above gives $v_i = \boxed{\left(\frac{M+m}{m}\right)\sqrt{\frac{gd^2}{2h}}}$.

Collisions, cont

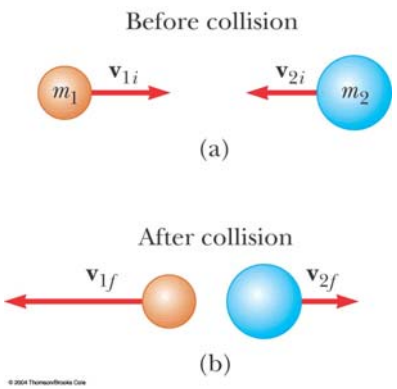
- In an ~~perfectly~~ inelastic collision, some kinetic energy is still lost, but the objects do not stick together
- Elastic and perfectly inelastic collisions are two extreme cases, most actual collisions fall in between these two types
- Momentum is conserved in any type of collisions

Elastic Collisions

- Both momentum and kinetic energy are conserved

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f}$$

$$\frac{1}{2}m_1\mathbf{v}_{1i}^2 + \frac{1}{2}m_2\mathbf{v}_{2i}^2 = \frac{1}{2}m_1\mathbf{v}_{1f}^2 + \frac{1}{2}m_2\mathbf{v}_{2f}^2$$



From: $\frac{1}{2}m_1\mathbf{v}_{1i}^2 + \frac{1}{2}m_2\mathbf{v}_{2i}^2 = \frac{1}{2}m_1\mathbf{v}_{1f}^2 + \frac{1}{2}m_2\mathbf{v}_{2f}^2$

we have $m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2)$, factoring both sides :

$\cancel{m_1(v_{1i} - v_{1f})}(v_{1i} + v_{1f}) = \cancel{m_2(v_{2f} - v_{2i})}(v_{2f} + v_{2i})$ ...(1)

From: $m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f}$

we have $\cancel{m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i})}$ ...(2)

Divide (1) by (2), we have $v_{1i} + v_{1f} = v_{2f} + v_{2i}$, or,

$v_{1i} - v_{2i} = -(v_{1f} - v_{2f})$ ...(3)

Equation (3) shows that the relative velocity of the two particles before collision, $v_{1i} - v_{2i}$, equals the negative of their relative velocity after the collision $-(v_{1f} - v_{2f})$.

Final velocities v_{1f} , v_{2f} . To solve for v_{2f}, we rearrange (3) and obtain: $v_{1f} = v_{2f} + v_{2i} - v_{1i}$, and substitute it in the equation of linear momentum to get v_{2f}.

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f}$$

$$m_2 v_{2f} = m_1 v_{1i} + m_2 v_{2i} - m_1 v_{1f}$$

$$m_2 v_{2f} = m_1 v_{1i} + m_2 v_{2i} - m_1(v_{2f} + v_{2i} - v_{1i})$$

$$(m1 + m_2)\, v_{2f} = (m_2 - m_1)\, v_{2i} + 2\, m_1 v_{1i}$$

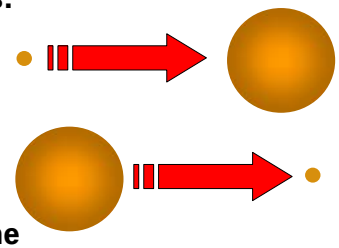
$$v_{2f} = \frac{(m_2 - m_1)\, v_{2i}}{(m1 + m_2)} + \frac{2\, m_1 v_{1i}}{(m1 + m_2)}$$

Similarly: $$v_{1f} = \frac{(m_1 - m_2)\, v_{1i}}{(m1 + m_2)} + \frac{2\, m_2 v_{2i}}{(m1 + m_2)}$$

Example. **Consider two elastic collisions:**

1) **a golf ball with speed *v* hits a stationary bowling ball head-on.**

2) **a bowling ball with speed *v* hits a stationary golf ball head-on.**



In which case does the golf ball have the greater speed after the collision?

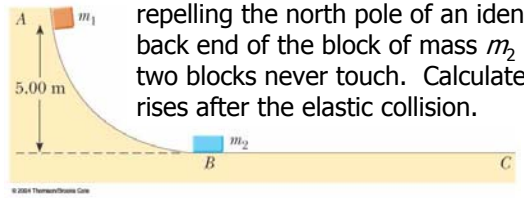
Answer:

The magnitude of the *relative velocity* has to be equal before and after the elastic collision! $v_{1i} - v_{2i} = -(v_{1f} - v_{2f})$

In case 1 the bowling ball will almost remain at rest due to its inertia, and the golf ball will bounce back with speed close to *v* (in its opposite direction).

In case 2 the bowling ball will keep going with speed close to *v* also due to its inertia. Hence the golf ball will move with speed close to 2*v*.

Example. Two blocks are free to slide along the frictionless wooden track *ABC* shown in Figure. A block of mass m_1 = 5.00 kg is released from *A*. Protruding from its front end is the north pole of a strong magnet, repelling the north pole of an identical magnet embedded in the back end of the block of mass m_2 = 10.0 kg, initially at rest. The two blocks never touch. Calculate the maximum height to which m_1 rises after the elastic collision.



Answer: Let v_1 be the speed of m_1 at B before collision.

$$\frac{1}{2}m_1 v_1^2 = m_1 g h$$

$$v_1 = \sqrt{2(9.80)(5.00)} = 9.90 \text{ m/s}$$

Let v_{1f} be the speed of m_1 at B after collision.

$$v_{1f} = \frac{(m_1 - m_2)\, v_{1i}}{(m1 + m_2)} + \cancel{\frac{2\, m_2 v_{2i}}{(m1 + m_2)}}$$

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_1 = -\frac{1}{3}(9.90) \text{ m/s} = -3.30 \text{ m/s}$$

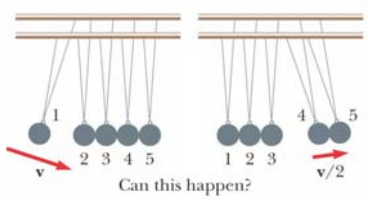
At the highest point (after collision)

$$m_1 g h_{max} = \frac{1}{2} m_1 (-3.30)^2 \qquad h_{max} = \frac{(-3.30 \text{ m/s})^2}{2(9.80 \text{ m/s}^2)} = \boxed{0.556 \text{ m}}$$

In the drawing on the right for Executive Stress Reliever, The initial total momentum is mv, and the final total momentum is

m(v/2) + m (v/2) = mv.

This obeys the conservation of linear momentum. Explain if the scenario shown is possible or not if the collision is elastic.



Answer:

Considering the energy aspect based on elastic collision. The initial energy in the system is $\frac{1}{2}mv^2$. The final energy in the system is $\frac{1}{2}m(v/2)^2 + \frac{1}{2}m(v/2)^2 = \frac{1}{4}mv^2$. This has violated the conservation of energy in elastic collision. So the above scenario is not possible.

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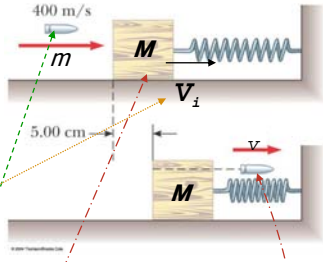


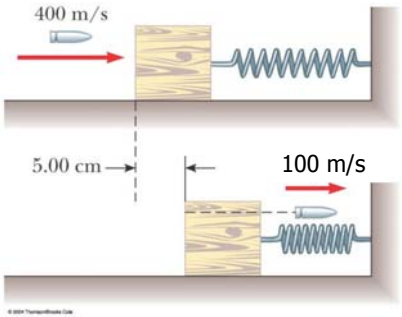
Answer:

Assume for an approximation that the bullet passes the block at a negligible time interval, and after the hit the block quickly reaches its maximum velocity V_i, and the bullet keeps going from the block penetration with a constant velocity *v*. The block then compresses the spring for a distance of 5.00 cm.

$$\frac{1}{2}M V_i^2 = \frac{1}{2}kx^2 \quad \Rightarrow \quad V_i = \sqrt{\frac{(900 \text{ N/m})(5.00 \times 10^{-2} \text{ m})^2}{1.00 \text{ kg}}} = 1.50 \text{ m/s}$$

Next by conservation of linear momentum, $m V_i = M V_i + m v$

$$v = \frac{m V_i - M V_i}{m} = \frac{(5.00 \times 10^{-3} \text{ kg})(400 \text{ m/s}) - (1.00 \text{ kg})(1.50 \text{ m/s})}{5.00 \times 10^{-3} \text{ kg}} = 100 \text{ m/s}$$



(b) the mechanical energy converted into internal energy in the collision.

Kinetic energy in bullet

Potential energy in spring

Answer:

Final energy: $\frac{1}{2}\left(5.00\times10^{-3}\ \text{kg}\right)\left(100\ \text{m/s}\right)^2+\frac{1}{2}\left(900\ \text{N/m}\right)\left(5.00\times10^{-2}\ \text{m}\right)^2$

Initial energy (Kinetic energy in bullet): $\frac{1}{2}\left(5.00\times10^{-3}\ \text{kg}\right)\left(400\ \text{m/s}\right)^2$

Final energy - Initial energy: 25 J + 1.125 J – 400 J

= -373.875 J

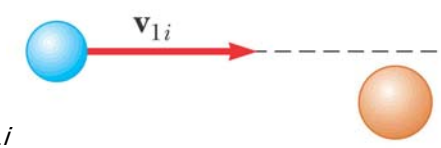
374 J was lost. The lost energy is mainly converted to heat.

Two-Dimensional Collisions

- The momentum is conserved in all directions
- Use subscripts for
  - identifying the object
  - indicating initial or final values
  - the velocity components
- If the collision is elastic, use conservation of kinetic energy as a second equation

Two-Dimensional Collision, example

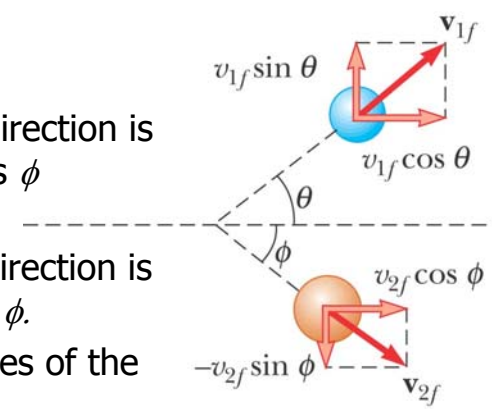
- Particle 1 is moving at velocity $\mathbf{v}_{1i}$ and particle 2 is at rest
- In the *x*-direction, the initial momentum is $m_1 v_{1i}$
- In the *y*-direction, the initial momentum is 0



(a) Before the collision

Two-Dimensional Collision, example cont

- After the collision, the momentum in the ***x***-direction is $m_1 v_{1f}\cos\theta + m_2 v_{2f}\cos\phi$
- After the collision, the momentum in the ***y***-direction is $m_1 v_{1f}\sin\theta + m_2 v_{2f}\sin\phi$.
- Let the angle take cares of the signs.



(b) After the collision

Problem-Solving Strategies – Two-Dimensional Collisions

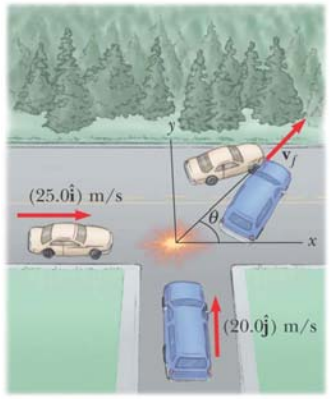
- If the collision is inelastic, kinetic energy of the system is not conserved, and additional information is needed
- If the collision is perfectly inelastic (the objects stick together), the final velocities of the two objects are equal (both objects are together). Solve the momentum equations for the unknowns.

Problem-Solving Strategies – Two-Dimensional Collisions

- If the collision is elastic, the kinetic energy of the system is conserved
- Equate the total kinetic energy before the collision to the total kinetic energy after the collision to obtain more information on the relationship between the velocities

Two-Dimensional Collision Example

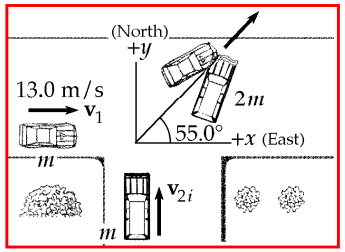
- Before the collision, the car has the total momentum in the *x*-direction only, and the van has the total momentum in the *y*-direction only
- After the collision, both have *x*- and *y*-components



Example. Two automobiles of equal mass approach an intersection. One vehicle is traveling with velocity 13.0 m/s toward the east and the other is traveling north with speed v_{2i}. Neither driver sees the other. The vehicles collide in the intersection and stick together, leaving parallel skid marks at an angle of 55.0° north of east. The speed limit for both roads is 35 mi/h and the driver of the northward-moving vehicle claims he was within the speed limit when the collision occurred. Is he telling the truth?

Answer:

We use conservation of momentum for the system of two vehicles for both northward and eastward components.



For the eastward direction:

$$M\,(13.0\ \text{m/s}) = 2M\,V_f \cos 55.0°$$

For the northward direction:

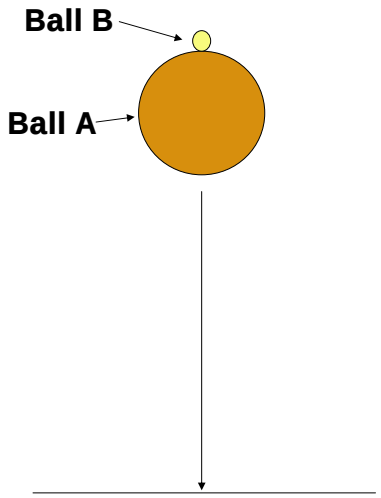
$$M\,v_{2i} = 2M\,V_f \sin 55.0°$$

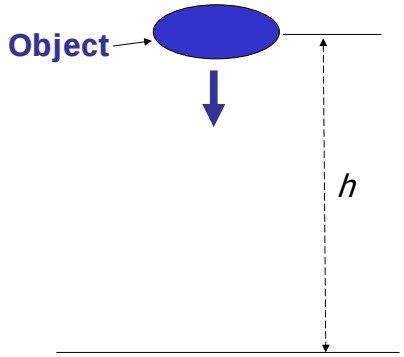
Divide the northward equation by the eastward equation to find:

$$v_{2i} = (13.0\ \text{m/s}) \tan 55.0° = 18.6\ \text{m/s} =$$ 18.6 x3600/1609 = 41.62 mi/h

Thus, the driver of the north bound car was untruthful.

A basket ball (Ball A) of mass 500g and a table tennis ball (Ball B) of mass 20g are placed as shown in figure. The two balls are released at a certain height at the same time with the table tennis ball on top of the basket ball. What can you observe and why?





Will this object (pumped with high pressure air) bounce back with a height greater than *h*?

Why?

The Center of Mass

The Center of Mass

- There is a special point in a system or object, called the ***center of mass***, that moves as if all of the mass of the system is concentrated at that point
- The system will move as if an external force were applied to a single particle of mass *M* located at the center of mass
  - *M* is the total mass of the system

Center of Mass, Coordinates

- The coordinates of the center of mass are

$$x_{CM} = \frac{\sum_i m_i x_i}{M} \quad y_{CM} = \frac{\sum_i m_i y_i}{M} \quad z_{CM} = \frac{\sum_i m_i z_i}{M}$$

  - where M is the total mass of the system

Center of Mass, position

- The center of mass can be located by its position vector, $\mathbf{r}_{CM}$

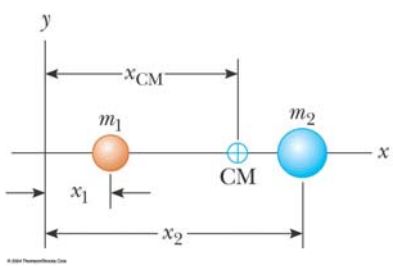
$$\mathbf{r}_{CM} = \frac{\sum_i m_i \mathbf{r}_i}{M}$$

- $\mathbf{r}_i$ is the position of the ith particle, defined by

$$\mathbf{r}_i = x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}$$

Center of Mass, Example

- Both masses are on the x-axis
- The center of mass is on the x-axis
- The center of mass is closer to the particle with the larger mass



Center of Mass, Extended Object

- Think of the extended object as a system containing a large number of particles
- The separation of particle is small, so the mass can be considered a continuous mass distribution

Center of Mass, Extended Object, Coordinates

- The coordinates of the center of mass of the object are

$$x_{CM} = \frac{1}{M}\int x\,dm \quad y_{CM} = \frac{1}{M}\int y\,dm$$

$$z_{CM} = \frac{1}{M}\int z\,dm$$

Center of Mass, Extended Object, Position

- The position of the center of mass can also be found by:

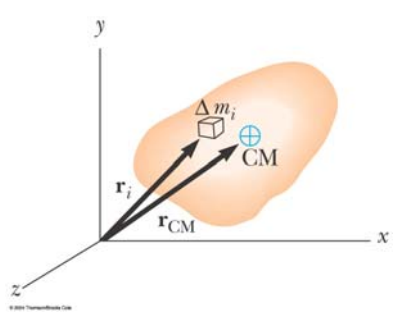
$$\mathbf{r}_{CM} = \frac{1}{M}\int \mathbf{r}\,dm$$

where r is a vector

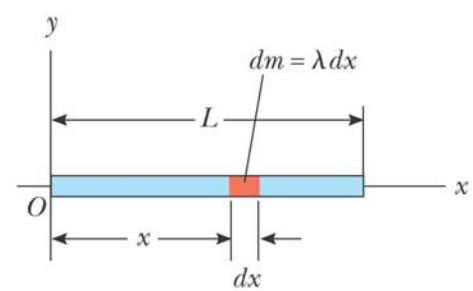
- The center of mass of any symmetrical object lies on an axis of symmetry and on any plane of symmetry

Center of Mass, Example

- An extended object can be considered a distribution of small mass elements, Δm
- The center of mass is located at position $\mathbf{r}_{CM}$



Example. Find the **center of mass** of a rod of mass *M* and length *L.*



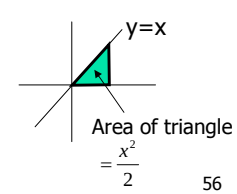
Answer:

Let λ (read as "lambda") denote the linear mass density, or the mass per unit length, then $\lambda = \frac{M}{L}$.

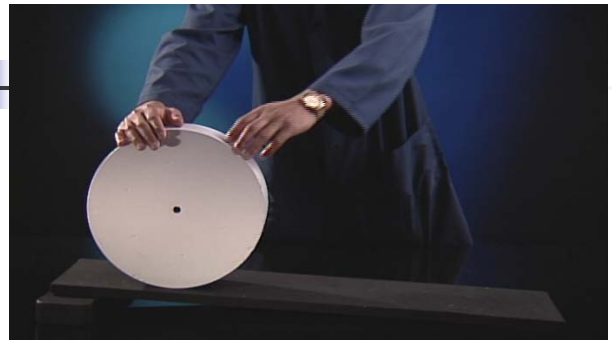
If we divide the rod into elements of length *dx*, then $\lambda = \frac{dm}{dx}$, or $dm = \lambda\, dx$.

$$X_{cm} = \frac{1}{M}\int_o^L x\,dm = \frac{1}{M}\int_o^L x\lambda\,dx$$

$$= \frac{\lambda}{M}\left[\frac{x^2}{2}\right]_o^L = \frac{\lambda L^2}{2M} = \frac{L}{2}$$



How to do this?



The disc is to roll down-hill from left to right, and to roll up-hill from right to left when it is released from rest.

How to do this?

Motion of a System of Particles

- Assume the total mass, *M*, of the system remains constant
- We can describe the motion of the system in terms of the velocity and acceleration of the center of mass of the system
- We can also describe the momentum of the system and Newton's Second Law for the system

Velocity and Momentum of a System of Particles

- The velocity of the center of mass of a system of particles is

$$\mathbf{v}_{\mathrm{CM}} = \frac{d\mathbf{r}_{\mathrm{CM}}}{dt} = \frac{\sum_i m_i \mathbf{v}_i}{M}$$

- The momentum can be expressed as $M\mathbf{v}_{\mathrm{CM}} = \sum_i m_i \mathbf{v}_i = \sum_i \mathbf{p}_i = \mathbf{p}_{\mathrm{tot}}$
- The total linear momentum of the system equals the total mass multiplied by the velocity of the center of mass

Acceleration of the Center of Mass

- The acceleration of the center of mass can be found by differentiating the velocity with respect to time

$$\mathbf{a}_{\mathrm{CM}} = \frac{d\mathbf{v}_{\mathrm{CM}}}{dt} = \frac{1}{M}\sum_i m_i \mathbf{a}_i$$

Forces In a System of Particles

- The acceleration can be related to a force $$M\mathbf{a}_{\mathrm{CM}} = \sum_i F_i$$
- If we sum over all the internal forces, they cancel in pairs, and the net force on the system is caused only by the external forces

Newton's Second Law for a System of Particles

- Since the only forces are external, the net external force equals the total mass of the system multiplied by the acceleration of the center of mass:

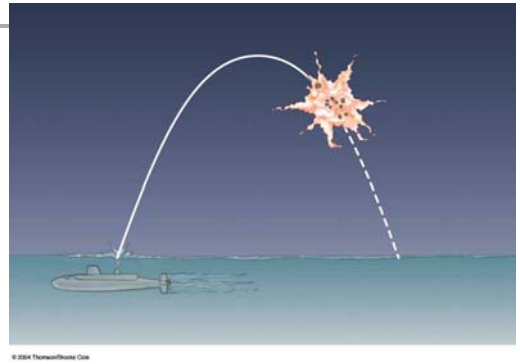
$$\Sigma\mathbf{F}_{\mathrm{ext}} = M\,\mathbf{a}_{\mathrm{CM}}$$

- The center of mass of a system of particles of combined mass M moves like an equivalent particle of mass M would move under the influence of the net external force on the system

Momentum of a System of Particles

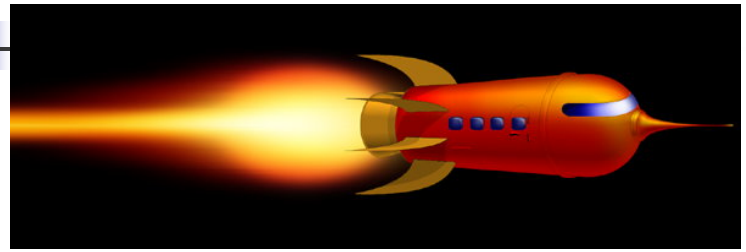
- The total linear momentum of a system of particles is conserved if no net external force is acting on the system
- $M\,\mathbf{v}_{\mathrm{CM}} = \mathbf{p}_{\mathrm{tot}}$ = constant when $\Sigma\mathbf{F}_{\mathrm{ext}} = 0$

Motion of the Center of Mass, Example



- A projectile is fired into the air and suddenly explodes
- With no explosion, the projectile would follow the dotted line
- After the explosion, the center of mass of the fragments still follows the dotted line, the same parabolic path the projectile would have followed with no explosion

Rocket Propulsion (Disk 2)

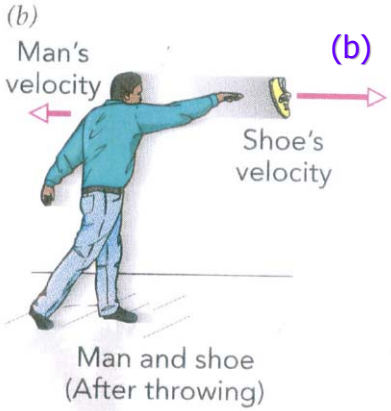


- The operation of a rocket depends upon the law of conservation of linear momentum as applied to a system of particles, where the system is the rocket plus its ejected fuel

Example. Image that you are sitting in the middle of a frozen lake with zero velocity and no momentum. It's a warm day and the wet ice is remarkably slippery. Try as you like, you can't seem to get moving at all. How do you get off the ice?

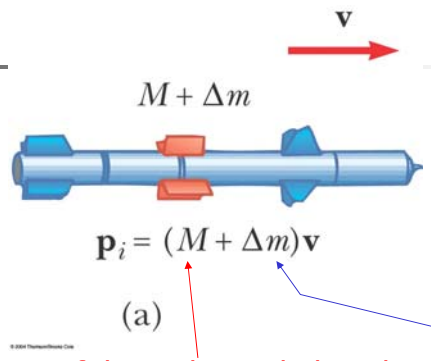
Answer:

(a) A man who is holding a shoe while standing still on ice has zero momentum.



(b) Once he has thrown the shoe to the right, the shoe has a momentum to the right, and the man has a momentum to the left. The total momentum of the man and shoe is still zero. Because the man is much more massive than the shoe, the shoe moves much faster than the man.

Rocket Propulsion, 2

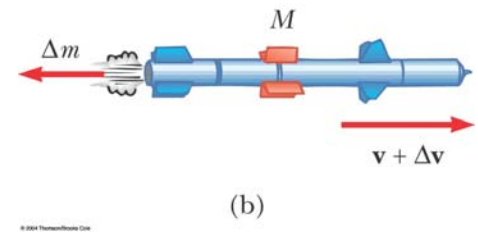


- The initial mass of the rocket including the major portion of the unburned fuel, plus, a very very very small amount of fuel to be burned immediately is $M + \Delta m$ at time t_i and velocity **v**
- The initial momentum of the system is

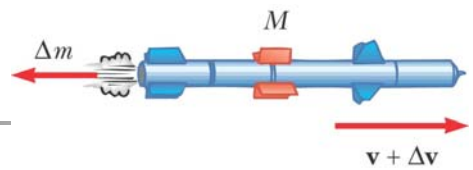
$$\mathbf{p}_i = (M + \Delta m)\,\mathbf{v}$$

Rocket Propulsion, 3

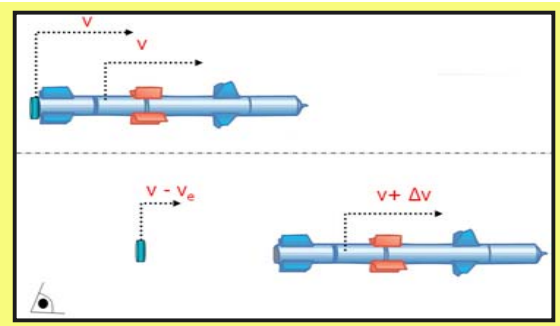
- At some time $t + \Delta t$, the rocket's mass has been reduced to M and an amount of fuel, Δm has been ejected
- The rocket's speed has then increased by $\Delta\mathbf{v}$



Rocket Propulsion, 4



- Because the gases are given some momentum when they are ejected out of the engine, the rocket receives a compensating momentum in the opposite direction
- Therefore, the rocket is accelerated as a result of the "push" (or thrust) from the exhaust gases
- In free space, the center of mass of the system (rocket plus expelled gases) moves uniformly, independent of the propulsion process



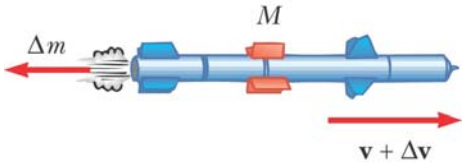
If the fuel is ejected with a speed v_e relative to rocket, the velocity of the ejected fuel with respect to the Earth is $v - v_e$.

By equating the total initial momentum of the system to the total final momentum, we have

$$(M + \Delta m)v = M(v + \Delta v) + \Delta m(v - v_e)$$

Simplifying the equation, we have $M\,\Delta v = v_e\,\Delta m$

But the increase in exhaust mass m is equal to the decrease mass M. So we have $\Delta m = -\Delta M$. This give $M\,\Delta v = v_e\,(-\Delta M)$.



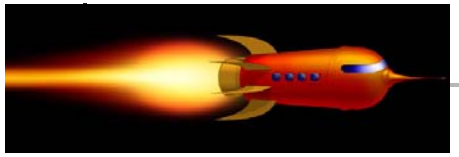
By integrating $M\,\Delta v = v_e(-\Delta M)$, we have

$$M\,dv = -v_e\,dM$$

$$\int_{v_i}^{v_f} dv = -v_e \int_{M_i}^{M_f} \frac{dM}{M}$$

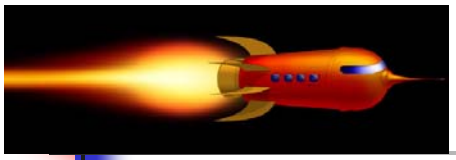
$$v_f - v_i = -v_e\,[\ln M]_{M_i}^{M_f}$$

$$v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right)$$



$$v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right)$$

- The increase in rocket speed is proportional to the speed of the escape gases (v_e)
  - So, the exhaust speed should be very high
- The increase in rocket speed is also proportional to the natural log of the ratio M_i/M_f
  - So, the ratio should be as high as possible, meaning the mass of the rocket should be as small as possible and it should carry as much fuel as possible, i.e., its payload (total weight – fuel weight) should be as small as possible.
  - **What if M_i/M_f is close to 1?**
  - **What if v_e is decreasing?**



$$M\,dv = -v_e\,dM$$

- The thrust on the rocket is the force exerted on it by the ejected exhaust gases

$$\text{Thrust} = M\frac{dv}{dt} = \left|v_e \frac{dM}{dt}\right|$$

- The thrust increases as the exhaust speed increases
- The thrust increases as the rate of change of mass increases
  - The rate of change of the mass is called the **burn rate**

Example. A size C5 model rocket engine has an average thrust of 5.26 N, a fuel mass of 12.7 grams, and an initial (total) mass of 25.5 grams. The duration of its burn is 1.90 s.

(a) What is the average exhaust speed of the engine?

(b) If this engine is placed in a rocket body of mass 53.5 grams, what is the final velocity of the rocket if it is fired in outer space? Assume the fuel burns at a constant rate.

Answer:

(a)

The fuel burns at a rate $\frac{dM}{dt} = \frac{12.7\text{ g}}{1.90\text{ s}} = 6.68\times10^{-3}\text{ kg/s}$

Thrust $= v_e\frac{dM}{dt}$: $5.26\text{ N} = v_e\left(6.68\times10^{-3}\text{ kg/s}\right)$

$v_e = \boxed{787\text{ m/s}}$

(b)

$v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right)$: $v_f - 0 = (\mathbf{787}\text{ m/s})\ln\left(\frac{53.5\text{ g}+25.5\text{ g}}{53.5\text{ g}+25.5\text{ g}-12.7\text{ g}}\right)$

$v_f = \boxed{138\text{ m/s}}$