Measurements and Data Analysis

1 Introduction

The central point in experimental physical science is the *measurement* of physical quantities. Experience has shown that all measurements, no matter how carefully made, have some degree of *uncertainty* that may come from a variety of sources. The study and evaluation of uncertainty in measurement is often called *uncertainty analysis* or *error analysis*.

The complete statement of a measured value should include an estimate of the level of confidence associated with the value. Properly reporting an experimental result along with its uncertainty allows other people to make judgements about the quality of the experiment and it facilitates meaningful comparisons with other similar values or a theoretical prediction. Without an estimated uncertainty, it would be impossible to answer the basic scientific questions such as: "Does my result agree with theoretical prediction or results from other experiments?" This question is fundamental for deciding if a scientific hypothesis is to be confirmed or refuted.

2 Uncertainties in Measurements

When we make a measurement, it is generally assumed that there exists a *true value* for any physical quantity being measured. Although we may never know this true value exactly, we attempt to discover it to the best of our ability with the time and resources available. It is expected that there will be some difference between the true value and the measured value. As we make measurements by different methods or even when making repeated measurements using the same method, we generally obtain different results. The correct way to state the result of a measurement is to give a best estimate of the quantity and the range within with we are confident the quantity lies. The most common way to show the range of values that we believe the true value lies in is:

$$\boxed{\text{measured value} = \text{best estimate} \pm \text{uncertainty}}$$

Suppose a physical quantity x is measured. The reliability of this value is indicated by giving an estimate of the possible uncertainty δx. The result of the measurement is then stated as

$$\boxed{(\text{measured value of } x) = x_{\text{best}} \pm \delta x}$$

This statement means, first, that the experimenter's best estimate for the quantity x is the number x_{best}, and second, that the experimenter is *reasonably* confident the quantity lies somewhere between $x_{\text{best}} - \delta x$ and $x_{\text{best}} + \delta x$. The number δx is called the *uncertainty* or *error* in the

measurement of x. If you measure x several times, your measurement in each trial should lie within the range.

It is very difficult to establish fixed rules for estimating the size of an uncertainty or the range. One can only say that the range must be large enough so that the true value (known value) of the measured quantity very likely lies within the range defined by the uncertainty limits, i.e. between $(x_{\text{best}} - \delta x)$ and $(x_{\text{best}} + \delta x)$. If it does, then your experimental value is equal to the true value within experimental uncertainty. Otherwise, your experimental value is not equal to the true value within the experimental uncertainties. If this happens, further investigation has to be carried out by suggesting and explaining specifically what caused the discrepancy.

3 Types of Experimental Uncertainties

Experimental uncertainty generally can be classified as being of two types: (1) *systematic uncertainties*, which increase or decrease all measurements of a quantity in the same sense (either all measurements will tend to be too large or tend to be too small); and (2) *random uncertainties*, which are completely random.

§3.1 Systematic Uncertainties

Systematic uncertainties are usually associated with particular measurement instruments or techniques, such as an improperly calibrated instrument or bias on the part of the experimenter. Conditions from which systematic uncertainties can result include:

1. An improperly "zeroed" instrument, for example, an ammeter gives non-zero reading even when there is no current through it. In this case, it would systematically give an incorrect reading larger than the true value.
2. A faulty instrument, such as a thermometer that reads 101°C when immersed in boiling water at standard atmospheric pressure. This thermometer is faulty because the reading should be 100°C.
3. Personal error, such as using a wrong constant in calculation or always taking a high or low reading of a scale division. Other examples of personal systematical uncertainty are shown in Figure 1. Reading a value from a scale generally involves lining up something, such as a mark on the scale. The alignment – and hence the value of the reading – will depend on the position of the eye (parallax).

Systematic uncertainties are hard to deal with as it depends on the skills of the experimenter to recognize the sources of such uncertainties and to prevent them. However, once determined, such uncertainties can be removed from the reported results. For example, one could in principle take a good quality ruler to the National Bureau of Standards and compare it to a very accurate standard meter stick there and find exactly how much larger or smaller a length it tends to read at a standard temperature. One could then correct the previous results with appropriate corrections.

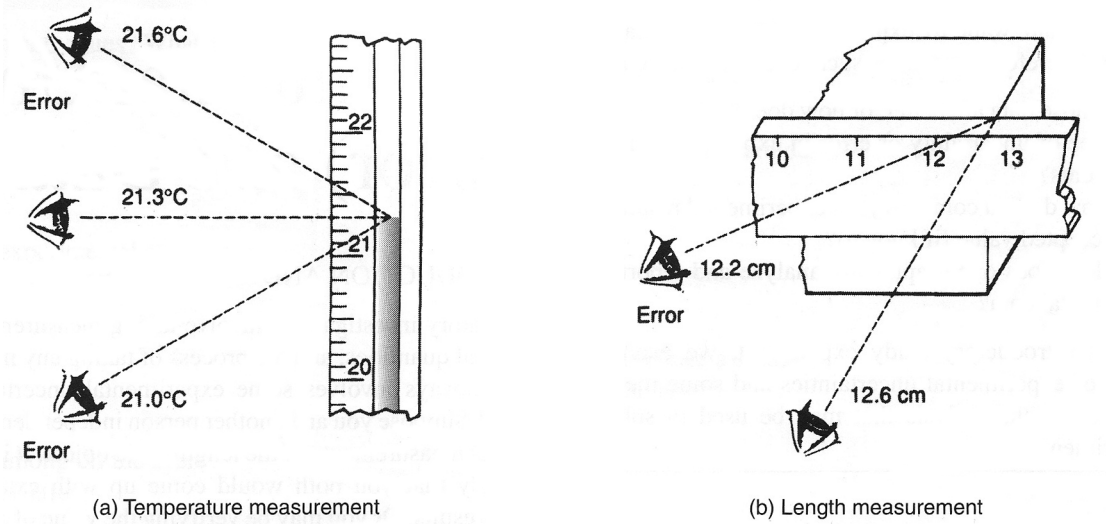

Figure 1: Examples of personal error due to parallax in reading (a) a thermometer and (b) a meter stick. Readings may systematically be made either too high or too low.

§3.2 *Random Uncertainties*

Random uncertainties result from a series of small, unknown and unpredictable variations that arise in all experimental measurement situations. Conditions in which random uncertainties can result include:

1. Unpredictable fluctuations in experimental conditions such as temperature.
2. Mechanical vibrations of an experimental setup.
3. Unbiased estimates of measurement readings by the experimenter. One such example is to time the revolution of a simple pendulum using stopwatch. Our reaction time in starting and stoping the stopwatch will vary. Since either possibility is equally likely, we will sometimes overestimate and sometimes underestimates if we repeat the measurement several times.

Repeated measurements with random uncertainties give slightly different values each time. In principle, random uncertainties can never be completely eliminated. However, unlike systematical uncertainties, there are definite statistical rules for estimating their magnitudes based on repeated measurements.

4 Accuracy and Precision

Accuracy and *precision* are commonly used synonymously, but in experimental measurements there is an important distinction. The accuracy of a measurement signifies how close it comes to the true (accepted) value – that is, how nearly correct it is.

Example: Two independent measurement results using the diameter d and circumference C of a circle in the determination of the value of π are 3.140 and 3.143. The second result is more accurate than the first because the true value of π, to four figures, is 3.142.

Precision refers to the agreement among repeated measurements – that is, the "spread" of the measurements or how close they are together. The more precise a group of measurements, the closer together they are. However, a large degree of precision does not necessarily imply accuracy as illustrated in Figure 2.

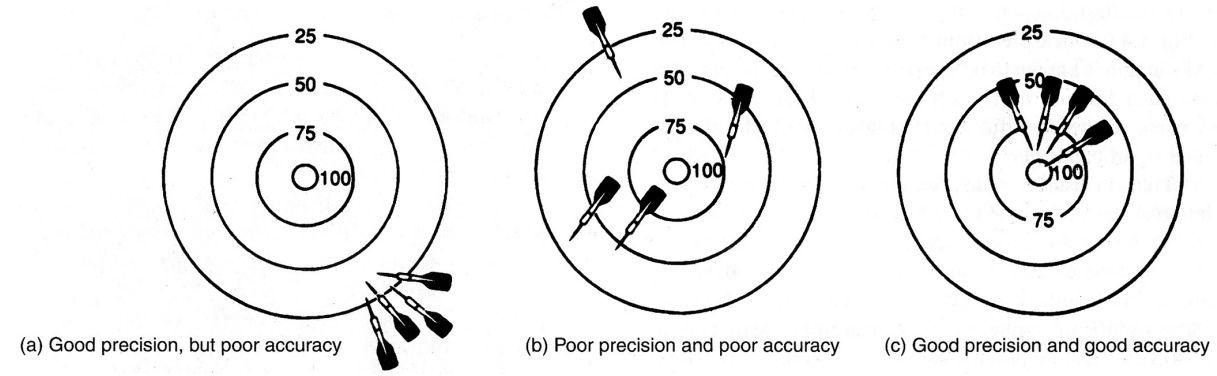

Figure 2: The true value in this analogy is the bull's eye. The degree of scattering is an indication of precision – the closer together a dart grouping, the greater the precision. A group (or symmetric grouping with an average) close to the true value represents accuracy.

Example: Two independent experiments give two sets of data with the expressed results and uncertainties of 2.5 ± 0.1 cm and 2.5 ± 0.2 cm respectively.

The first result is more precise than the second because the spread in the first measurements is between 2.4 and 2.6 cm, whereas the spread in the second measurements is between 2.3 and 2.7 cm. That is, the measurements of the first experiment are less uncertain than those of the second.

Obtaining *greater accuracy* for an experimental value depends in general on *minimizing systematic uncertainties*. Obtaining *greater precision* for an experimental value depends on *minimizing random uncertainties*.

5 Significant Figures

The number of *significant figures* means the number of digits known in some number for which the experimenter has confidence that they are correct. The significant figures are those digits that are reasonably known with the least significant digit having some uncertainty associated with it. The number of significant figures does not necessarily equal to the total digits in the number because zeroes are used as place keepers when digits are not known.

The rules for determining the number of significant figures in a number are:

- The most significant digit is the leftmost nonzero digit. In other words, zeroes at the left are never significant.
- In numbers that contain no decimal point, the rightmost nonzero digit is the least significant digit.
- In number that contains a decimal point, the rightmost digit is the least significant digit, regardless of whether it is zero or nonzero.
- The number of significant digits is found by counting the places from the most significant digit to the least significant digit.

Example: Consider the following list of numbers: (a) 3456; (b) 135700; (c) 0.003043; (d) 0.01000; (e) 1050.; (f) 1.034; (g) 0.0002608. All the numbers in the list have four significant figures for the following reasons: (a) there are four nonzero digits, they are all significant; (b) the two rightmost zeroes are not significant because there is no decimal point; (c) zeros at the left are never significant; (d) the zeroes at the left are not significant, but the three zeros at the right are significant because there is a decimal point; (e) there is a decimal point so all digits are significant; (f) again, there is a decimal point so all four are significant; (g) zeroes at the left are never significant.

When using numbers in calculations, care must be taken to keep track of the number of significant figures. The following rules should be used to determine the number of significant figures to retain at the end of a calculation:

- When adding or subtracting, figures to the right of the last column in which all figures are significant should be dropped.
- When multiplying or dividing, retain only as many significant figures in the result as are contained in the number with the least number of significant figures in the calculation.
- The last significant figure is increased by 1 if the figure beyond it (which is dropped) is 5 or greater.

These rules apply only to the determination of the number of significant figures in the final result. In the intermediate steps of a calculation, one more significant figure should be kept in order to have more accurate final result.

Example: Consider the addition of the following numbers:

$$753.1 + 37.08 + 0.697 + 56.3 = 847.177$$

Following the above rules strictly implies rewriting each number as shown below

$$753.1 + 37.1 + 0.7 + 56.3 = 847.2$$

in which the first digit beyond the decimal is the least significant digit. This is because that column is the rightmost column in which all digits are significant. It is noted that the first digit beyond the decimal is the last one that can be kept. Therefore 847.177 is rounded off to 847.2.

A similar process is used for multiplication and division. Consider the calculations below:

$$327.23 \times 36.73 = 12019.158 \qquad 327.23 \div 36.73 = 8.90906\,.$$

In each case, the result is rounded to four significant figures because the least significant number in each calculation (36.73) has only four significant figures. For the multiplication the result is 12020, and for the division it is 8.909.

We usually report a result with the best estimate for the quantity itself and the associated uncertainty. Generally, the uncertainty should almost always be rounded to one significant figure. The last significant figure reported for the best estimate of the result should usually be of the same order of magnitude (in the same decimal position) as the uncertainty.

Example: The mass of an object is determined to be 123.72 g and the corresponding uncertainty is determined to be 0.17 g. The mass of this object is then to be reported as (123.7 ± 0.2) g.

6 Absolute, Fractional and Percentage Uncertainties

Absolute uncertainty, δx, represents the actual amount by which the best estimated value is uncertain and has the same unit as the measured value itself. To perceive the significance of the uncertainty, we define a quantity called *fractional uncertainty*:

$$\text{Fractional Uncertainty} = \frac{\text{Absolute Uncertainty}}{\text{Best Estimated Value}} = \frac{\delta x}{x_{\text{best}}}$$

As the fractional uncertainty is a ratio of two quantities that have the same units, the fractional uncertainty itself has *no* units. Uncertainty is always expressed as a percentage of the best estimated value by multiplying the fractional uncertainty by 100% – *percentage uncertainty*.

Example: The length of an object is determined to be $L \pm \delta L = (2.3 \pm 0.1)$ cm. The absolute uncertainty in the length of the object is 0.1 cm. The fractional uncertainty is calculated as

$$\frac{\delta L}{L_{\text{best}}} = \frac{0.1\,\text{cm}}{2.3\,\text{cm}} = 0.044$$

and the percentage uncertainty is $0.044 \times 100\% = 4.4\%$. The final result can then be reported as $L = 2.3\,\text{cm} \pm 4\%$.

Fractional or percentage uncertainty is a better measure of precision than the absolute uncertainty. A smaller absolute uncertainty does not necessarily imply higher precision.

Example: Consider the following two measurements:

$L = (121.5 \pm 0.5)$ cm percentage uncertainty $= \dfrac{0.5\,\text{cm}}{121.5\,\text{cm}} = 0.4\%$

$L = (0.014 \pm 0.002)$ cm percentage uncertainty $= \dfrac{0.002\,\text{cm}}{0.014\,\text{cm}} = 10\%$

The second measurement has a smaller absolute uncertainty but larger percentage uncertainty. The first measurement is more precise because it has lower percentage uncertainty. If we use a more precise measuring device, such as vernier caliper, in the second measurement, we might have the following new measured value:

$L = (0.014 \pm 0.001)$ cm percentage uncertainty $= \dfrac{0.001\,\text{cm}}{0.014\,\text{cm}} = 7\%$

The percentage uncertainty has reduced from 10% to 7%. Clearly, we have increased the precision in this measurement by reducing the absolute uncertainty.

7 Estimating Uncertainties using Statistics

If we make repeated measurements of the same quantity, we can apply statistical analysis to study the uncertainties in our measurements. The uncertainties are determined from the data themselves without further estimates in statistical analysis. The important parameters in such analysis are the *mean*, the *standard deviation* and the *standard error*.

§7.1 Mean

If all sources of systematic uncertainties have been identified and reduced to a negligible level, statistical theory says that the *mean* is the best estimation to the true value. In formal mathematical terms, the mean value $\bar{x}$ of a set of N measurements is

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_N}{N} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

where the summation sign Σ is the shorthand notation indicating the sum of N measurements x_1 to x_N.

Example: Consider an experiment in which a small object falls through a fixed distance and the time for it to fall is measured using a stopwatch. The table below shows ten values recorded for the time of fall.

Time (s)	0.64	0.61	0.63	0.53	0.59	0.65	0.60	0.61	0.64	0.71

Table: Times for object to fall through a fixed distance.

We might have hoped that on each occasion we measured the time of fall of the object, we would have obtained the same value. Unfortunately, this is not true in the case of the above experiment and it is, in general, untrue in any experiment.

We could expect the time that it really took for the object fall to lie somewhere between the two extreme measured values, namely between 0.53 s and 0.71 s. If a single value for the time of fall is required, we can do no better than to calculate the mean of the ten measurements that were made. The mean time is given by

$$\bar{t} = \frac{(0.64 + 0.61 + 0.63 + 0.53 + 0.59 + 0.65 + 0.60 + 0.61 + 0.64 + 0.71)\,\text{s}}{10}$$
$$= \frac{6.21\,\text{s}}{10} = 0.621\,\text{s}$$

It is difficult to determine the number of significant figures to be used when quoting the mean unless we have an estimate for the uncertainty in the mean value.

§7.2 Standard Deviation

Having obtained a set of measurements and determined the mean value, it is helpful to report how widely the individual measurements are scattered from the mean. A quantitative description of this scatter of measurement will give an idea of the precision of the experiment.

Statistical theory states that the precision of the measurement can be determined by calculating a quantity called the *standard deviation* from the mean of the measurements. The standard deviation from the mean, which has a symbol of σ, is defined as

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

The standard deviation answers the question "If, after having made a series of measurement and averaging the results, I now make one more measurement, how close is the new measurement likely to come to the previous mean?"

The quantity σ gives the probability that the measurements fall within a certain range of the measured mean. Probability theory states that approximately 68.3% of all repeated measurements should fall within a range of plus or minus σ from the mean. Furthermore, 95.5% of all repeated measurements should fall within a range of 2σ from the mean. As a final note, 99.73% of all measurements should fall within 3σ of the mean. This implies that if one of the measurements is 3σ or farther from the mean, it is very unlikely that it is a random uncertainty.

Example: Consider an experiment in which the period of a simple pendulum (time for one complete swing) is measured using a stopwatch. The table below shows ten repeat timings of the simple pendulum.

Time (s)	5.8	6.1	5.9	5.5	5.7	5.8	6.0	5.9	5.9	5.6

Table: Ten repeat measurements of period for a simple pendulum.

The mean time is calculated as 5.82 s and the standard deviation from the mean is 0.181 s. Except for these measurements 6.1 s, 5.5 s and 5.6 s, the remaining measurements (7 out of 10) are within the range of plus or minus σ from the mean. This is consistent with the statistical theory that approximately 68% should fall within $\pm\sigma$ from the mean.

We can regard the standard deviation as an estimate of the uncertainty for any *single* measurement in the table. For instance, 0.2 s can treated as the uncertainty of the second measurement 6.1 s and the result can be reported as (6.1 ± 0.2) s.

If another measurement (the eleventh reading) is to be taken under the same conditions, it is expected that the new measurement would approximately have a 68.3% probability of falling within the range of (5.8 ± 0.2) s.

§7.3 Standard Error

To get a good estimate of some quantity, several repeated measurements have to be taken. Another issue that can be addressed by these repeated measurements is the precision of the mean. After all, this is what is really of concern, because the mean is the best estimate of the true value. The precision of the mean is indicated by a quantity called the *standard error*. Other common name is *standard deviation of the mean*.

Once the standard deviation has been calculated for N measurements, it is very easy to calculate standard error (which has a symbol of α) using the equation

$$\boxed{\alpha = \frac{\sigma}{\sqrt{N}}}$$

It is this quantity that answers the question, "If I repeat the entire series of N measurements and get a second mean, how close can I expect this second mean to come to the first one?" One needs not repeat the entire experiment multiple times to get the standard error. If the variation in data seem to be caused by a series of small random events, then the above equation can be used to find the standard error.

Example: In a fluid-flow experiment, 10 measurements were made of the volume of water flowing through the apparatus in one minute. The table below shows the data collected.

Volume of liquid collected (mL)									
33	45	43	42	45	42	41	44	40	42

Table: Volume of water collected in a fluid-flow experiment

The mean of the values in the above table is 41.7 mL with a standard deviation of 3.29 mL. The standard error is calculated as

$$\alpha = \frac{\sigma}{\sqrt{N}} = \frac{3.29\,\text{mL}}{\sqrt{10}} = 1.04\,\text{mL}$$

The best estimate of the volume of water collected in this experiment based on ten repeated measurements is (42 ± 1) mL. It is expected that the true volume of water collected would have a 68.3% probability of falling within the range of (42 ± 1) mL.

An important feature of the standard error is the factor $\sqrt{N}$ in the denominator. The standard deviation σ represents the average uncertainty in the individual measurements. Thus, if we were to make some more measurements (using the same technique), the standard deviation would not change appreciably. However, the standard error would slowly decrease as expected. If we make more measurements before computing the mean, we would naturally expect the final result to be more reliable. This is one of the obvious ways to improve the precision of our measurements.

8 Combining Uncertainties

Most physical quantities usually cannot be measured in a single direct measurement but are instead found in two distinct steps. First, we measure one or more quantities that can be measured directly and from which the quantity of interest can be calculated. Second, we use the measured values of these quantities to calculate the quantity of interest itself. For example, speed is often determined by measuring a distance traveled and the time taken to travel that distance, and then dividing the distance by the time.

When a measurement involves these two steps, the estimation of uncertainties also involves two steps. We must first estimate the uncertainties in the quantities measured directly and then determine how these uncertainties "propagate" through the calculations to produce an uncertainty in the final answer.

Suppose we have measured one or more quantities $x, y, \ldots, z$ with corresponding uncertainties $\delta x, \delta y, \ldots, \delta z$ and the measured values are used to compute the quantity of real interest, $q(x, \ldots, z)$. It is intuitive to see that the best estimated value for the quantity of interest q will be given by

$$\boxed{q_{\text{best}} = q(\bar{x}, \bar{y}, \ldots, \bar{z})}$$

If the uncertainties in $x, y, \ldots, z$ are *independent* and *random*, statistical theory states that the uncertainty of q is given by

$$\boxed{\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\,\delta x\right)^2 + \left(\frac{\partial q}{\partial y}\,\delta y\right)^2 + \cdots + \left(\frac{\partial q}{\partial z}\,\delta z\right)^2}}$$

Independent uncertainties means that the size of the uncertainty does not directly affect the size of the uncertainty in any of the other quantities.

Those who have studied calculus can recognize the symbol $\partial q/\partial x$ as the partial derivative of q with respect to x. The other symbols are defined in a similar manner. It is understood that some students have studied calculus, and some have not. Those who have studied calculus should be able to derive an equation for δq for any function needed. Those who have not studied calculus should not be intimidated its use here. It is only used to show the origin of these equations for the benefit of those who can use calculus to derive the appropriate equation for other cases of interest. In practice, for PC1221 and PC1222 students, any such equations needed will be given in a form that only requires the use of algebra.

The table below lists down the uncertainty propagation formulas for some of the common functions. Here, m and n are constants.

Addition	$q = mx + ny$	$\delta q = \sqrt{(m\,\delta x)^2 + (n\,\delta y)^2}$
Subtraction	$q = mx - ny$	$\delta q = \sqrt{(m\,\delta x)^2 + (n\,\delta y)^2}$
Multiplication	$q = mxy$	$\frac{\delta q}{q} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2}$
Division	$q = m\frac{x}{y}$	$\frac{\delta q}{q} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2}$
Power	$q = x^m y^n$	$\frac{\delta q}{q} = \sqrt{\left(m\,\frac{\delta x}{x}\right)^2 + \left(n\,\frac{\delta y}{y}\right)^2}$
Logarithm	$q = \ln(mx)$	$\delta q = \frac{\delta x}{\lvert x\rvert}$
Exponential	$q = \mathrm{e}^{mx}$	$\frac{\delta q}{q} = m\,\delta x$

Table: List of uncertainty propagation formulas for some common functions.

Example: Consider an experiment in which you are asked to calculate the volume V of a cylinder of diameter d and length ℓ. The volume of the cylinder is calculated as $\pi d^2\ell/4$. Using the uncertainty propagation formulas for multiplication, power and division, we have

$$\frac{\delta V}{V} = \sqrt{\left(\frac{2\,\delta d}{d}\right)^2 + \left(\frac{\delta \ell}{\ell}\right)^2}$$

Suppose that you obtained the following data:

$$\begin{aligned} d &= 12.32 \pm 0.02\,\mathrm{mm} \\ \ell &= 33.46 \pm 0.02\,\mathrm{mm} \end{aligned}$$

The best estimated volume of the cylinder is calculated as

$$V = \frac{\pi\left(12.32\times10^{-3}\right)^2\left(33.46\times10^{-3}\,\mathrm{m}\right)}{4} = 398.88\times10^{-8}\,\mathrm{m}^3$$

The uncertainty in the volume is

$$\begin{aligned} \frac{\delta V}{V} &= \sqrt{\left(2\,\frac{0.02\times10^{-3}\,\mathrm{m}}{12.32\times10^{-3}\,\mathrm{m}}\right)^2 + \left(\frac{0.02\times10^{-3}\,\mathrm{m}}{33.46\times10^{-3}\,\mathrm{m}}\right)^2} = 0.0033 \\ \Rightarrow \qquad \delta V &= 0.0033\times398.88\times10^{-8}\,\mathrm{m}^3 = 1.3163\times10^{-8}\,\mathrm{m}^3 \end{aligned}$$

The experimental volume of the cylinder is then reported as

$$V = (3.99 \pm 0.01)\times10^{-6}\,\mathrm{m}^3$$

---

9 Linear Least Squares Fits

One of the most common types of experiment involves the measurement of several values of two different variables to investigate the mathematical relationship between the two variables. Probably the most important experiments of this type are those for which the expected relation is *linear*. In mathematical terms, we will consider any two physical variables x and y that we suspect are connected by a linear relation of the form

$$\boxed{y = mx + c}$$

where m and c are constants.

If the two variables y and x are linearly related as above, then a graph of y against x should be a straight line that has gradient m and intersects the y axis at $y = c$. If we were to measure N different values $x_1, \ldots, x_N$ and the corresponding values $y_1, \ldots, y_N$ and if our measurements were subject to no uncertainties, then each of the points (x_i, y_i) would lie exactly on the line $y = mx + c$. In practice, there are uncertainties and the most we can expect is that the distance of each point (x_i, y_i) from the line will be reasonable compared with the uncertainties.

When we make a series of measurements, the problem is to find the straight line that best fits the measurements, that is, to find the best estimates for the constants m and c based on the data $(x_1, y_1), \ldots, (x_N, y_N)$. This problem can be approached graphically by drawing the best straight line possible through the data points. It can also be approached analytically, by means of minimizing the sum of the squares of the distance of each point from the line. This analytical method of finding the best straight line to fit a series of experimental points is called *linear regression*, or the *linear least squares fits* for a line.

The aim of the process is to determine the values of m and c that will produce the best straight line to the data. Any choice of values for m and c will produce a straight line, with values of y determined by the choice of x. For any such straight line (determined by a given m and c) there will be a deviation between each of the measured y's and the y's from the straight line fit at the value of the measured x's. The least squares fits is that m and c for which the sum of the squares of these deviations is a minimum. Statistical theory states that the appropriate values of m and c that will produce this minimum sum of squares of the deviations are given by the following equations:

$$\boxed{m = \frac{N \sum_{i=1}^{N} x_i y_i - \left(\sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} y_i\right)}{\Delta}} \quad \text{and} \quad \boxed{c = \frac{\left(\sum_{i=1}^{N} y_i\right)\left(\sum_{i=1}^{N} x_i^2\right) - \left(\sum_{i=1}^{N} x_i y_i\right)\left(\sum_{i=1}^{N} x_i\right)}{\Delta}}$$

where

$$\boxed{\Delta = N \sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2}$$

Based on the measured points, the best estimate for the uncertainty in the measurements of y is

$$\sigma_y = \sqrt{\frac{1}{N-2}\sum_{i=1}^{N}(y_i - mx_i - c)^2}$$

The uncertainties in m and c are

$$\sigma_m = \sigma_y\sqrt{\frac{N}{\Delta}} \quad \text{and} \quad \sigma_c = \sigma_y\sqrt{\frac{\sum_{i=0}^{N} x_i^2}{\Delta}}$$

These results were based on the assumptions that the measurements of y were all equally uncertain and that any uncertainties in x were negligible.

At this point, the best possible straight line fit to the data has been determined by the least squares fit process. There is, however, a quantitative measure of how well the data follow the straight line obtained by the least squares fit. It is given by the value of a quantity called the *correlation coefficient* or r. This quantity is a measure of the fit of the data to a straight line with $r = 1.000$ exactly signifying a perfect correlation, and $r = 0$ signifying no correlation at all. The equation to calculate r in terms of the general variables x and y is given by

$$r = \frac{N\sum_{i=1}^{N} x_i y_i - \left(\sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} y_i\right)}{\sqrt{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2}\sqrt{N\sum_{i=1}^{N} y_i^2 - \left(\sum_{i=1}^{N} y_i\right)^2}}$$

When performing a least squares fit to data, particularly when a small number of data points are involved, there is some tendency to obtain surprisingly good value for r even for data that do not appear to be very linear. For such cases, there is a 0.1% probability of obtaining a value of (i) $r \geq 0.992$ for $N = 5$; (ii) $r \geq 0.974$ for $N = 6$; (iii) $r \geq 0.951$ for $N = 7$; and (iv) $r \geq 0.925$ for $N = 8$ respectively, even though the data are not correlated. In practice, one can conclude that the data are very strong evidence for a linear relationship between the variables if the data produces value of r greater than the 0.1% probability for the particular value of N.

Example: Consider the set of data in the following table that was taken by measuring the coordinate position z of some object as a function of time t.

t (s)	1.00	2.00	3.00	4.00	5.00
z (m)	7.57	11.97	16.58	21.00	25.49

Table: Coordinate of some object z as a function of time t.

The question to be answered is whether or not the data are consistent with constant velocity. It is assumed that $z \neq 0$ at $t = 0$. If the speed v is constant, the data can be fitted by an equation of the form

$$z = vt + z_0$$

Thus, v will be the gradient of a graph of z versus t and z_0 will be the y-intercept, which is the coordinate position at the arbitrarily chosen time $t = 0$.

The gradient and y-intercept are calculated as

$$v = \frac{N\sum_{i=1}^{N} t_i z_i - \left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} z_i\right)}{N\sum_{i=1}^{N} t_i^2 - \left(\sum_{i=1}^{N} t_i\right)^2} = 4.487\,\mathrm{m/s}$$

$$z_0 = \frac{\left(\sum_{i=1}^{N} z_i\right)\left(\sum_{i=1}^{N} t_i^2\right) - \left(\sum_{i=1}^{N} t_i z_i\right)\left(\sum_{i=1}^{N} t_i\right)}{N\sum_{i=1}^{N} t_i^2 - \left(\sum_{i=1}^{N} t_i\right)^2} = 3.061\,\mathrm{m}$$

The uncertainties of the gradient and y-intercept are found to be $\sigma_m = 0.0165\,\mathrm{m/s}$ and $\sigma_c = 0.0548\,\mathrm{m}$ respectively. Thus, the velocity is determined to be $(4.49 \pm 0.02)\,\mathrm{m/s}$ and the coordinate at $t = 0$ is found to be $(3.06 \pm 0.05)\,\mathrm{m}$.

The data points (t, z)'s and the fitted straight line can then be plotted (see Figure 3). The best estimate for the uncertainty of dependent variable $\sigma_y = 0.05\,\mathrm{m}$ can be used as an estimation for the vertical error-bar of each data point. We can visually conclude whether the data are correlated by checking the fitted straight line can pass through most of the data points within error-bars. Quantitatively, the correlation coefficient is calculated by

$$r = \frac{N\sum_{i=1}^{N} t_i z_i - \left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} z_i\right)}{\sqrt{N\sum_{i=1}^{N} t_i^2 - \left(\sum_{i=1}^{N} t_i\right)^2}\sqrt{N\sum_{i=1}^{N} z_i^2 - \left(\sum_{i=1}^{N} z_i\right)^2}} = 0.999959$$

We can then conclude that the data shows an almost perfect linear relationship since r is so close to 1.000.

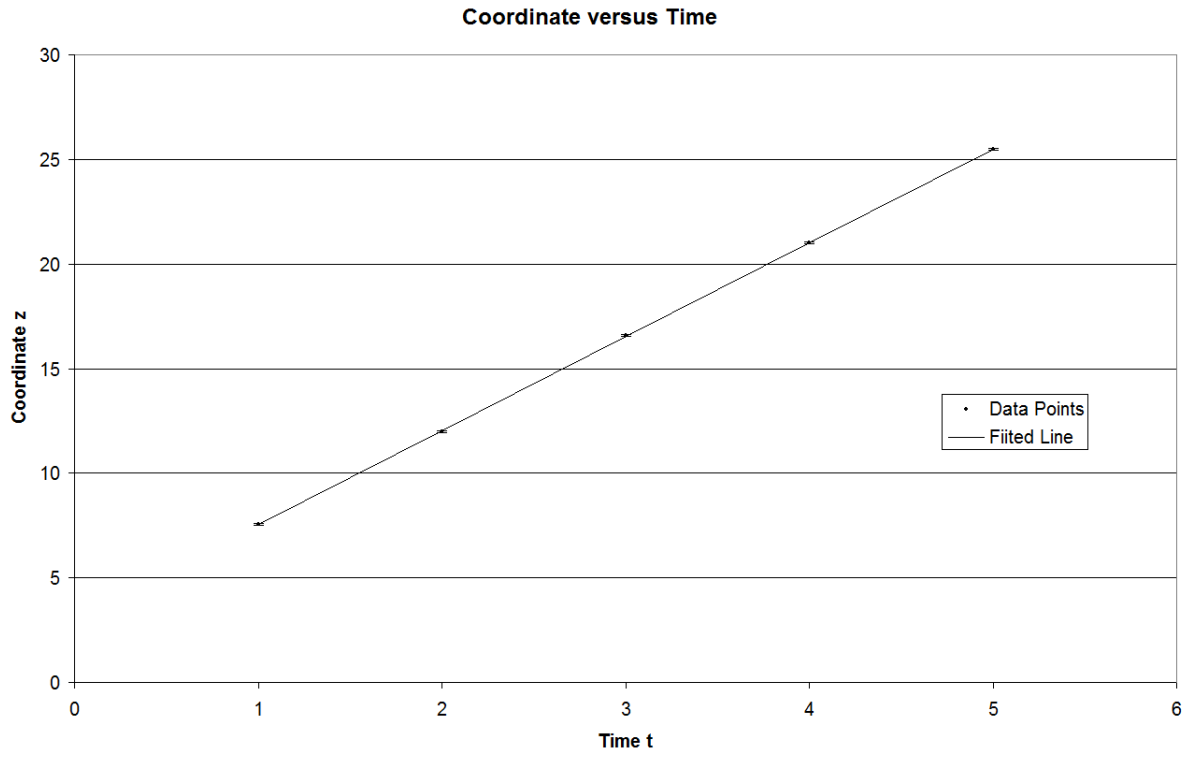

Figure 3: Graph of coordinate z versus time t.

10 Comparing Results

When our results are subjected to experimental uncertainties, we cannot expect exact equality when comparing a measurement with a known value or with another measurement. Instead, we have to take the uncertainty ranges into account while making the comparison.

In some experiments, the true value of the quantity (accepted value or theoretical value) being measured will be considered to be known. In those cases, the accuracy of the experiment will be determined by comparing the experimental value, $E \pm \delta E$, with the known value. If the true value *lies within* the range $E \pm \delta E$, we regard the experimental value equal to the true value within experimental uncertainties. Quantitatively, we can calculate the *percentage discrepancy* of the experimental value compared to the true value defined as:

$$\text{Percentage Discrepancy} = \frac{|\text{Experimental Value} - \text{True Value}|}{\text{True Value}} \times 100\%$$

If the percentage discrepancy is less than 10%, we can conclude that the experimental value is equal to the true value within 10% experimental uncertainty.

In other cases, a given quantity will be measured by two different methods. There will then be two different experimental values, $E_1 \pm \delta E_1$ and $E_2 \pm \delta E_2$, but the true value may not be

known. For this case, we check whether the range $E_1 \pm \delta E_1$ has any *overlap* with the range $E_2 \pm \delta E_2$. These two experimental values are equal within the uncertainties if there is some overlap. Quantitatively, *percentage difference* between the two experimental values will be calculated. Note that this tells nothing about the accuracy of the experiment, but will be a measure of the precision. The percentage difference between the two measurements is defined as

$$\text{Percentage Difference} = \frac{|E_2 - E_1|}{|E_1 + E_2|/2} \times 100\%$$

Here, the absolute difference between the two measurements has been compared with the average of the two measurements. The average is chosen as the basis for comparison when there is no reason to think that one of the measurements is any more reliable than the other. If the percentage difference is less than 10%, we can conclude that these two experimental values are equal within 10% experimental uncertainty.

References

[1] "Experimental methods: An introduction to the analysis and presentation of data," Les Kirkup, John Wiley.

[2] "An introduction to error analysis: The study of uncertainties in physical measurements," John R. Taylor, University Science Books.

[3] "Data analysis with Excel: An introduction for physical sciences," Les Kirkup, Cambridge University Press.

[4] "Experimentation: An introduction to measurement theory and experimental design," D. C. Baird, Prentice Hall.

[5] "An introduction to uncertainty in measurement," Les Kirkup and Bob Frenkel, Cambridge University Press.

[6] "Data reduction and error analysis for the physical sciences," Philip R. Bevington and D. Keith Robinson, Mc Graw Hill.

Last updated: Wednesday 4th August, 2010 12:00am (KHCM)