CZ3102 Scientific Modeling

Tutorial/Home Work Set 1

for week 3 (starting 19 Jan 2004)

1.1 Mathematical Background, Taylor series expansion

This and the next few questions are testing and strengthing your background in linear algebra and calculus. You are supposed to have learnt them in other modules.

(a) Expand the one-variable function $(a+x)^{-2}$ to $O(x^3)$ around $x=0$.

(b) Expand the function $1/x$ around $x=1$ to $O((x-1)^3)$.

(c) Expand the multi-variable function $1/|\mathbf{a}-\mathbf{r}|$ for small $\mathbf{r}$ to linear terms in $\mathbf{r}$ (so that the errors are of quadratic in $\mathbf{r}$, i.e., expand to $O(\mathbf{r}^2)$, where $\mathbf{a}=(a_1,a_2,a_3)$ and $\mathbf{r}=(x,y,z)$ are three-dimensional vectors, and the absolute sign $|\cdots|$ means Euclidean norm (3D distance).

1.2 Linear Least Squares

Solve the set of equations

$$
\begin{aligned}
x+0y &= 1, && (1)\\
0x+y &= 2, && (2)\\
x+y &= 4, && (3)\\
2x+y &= 5. && (4)
\end{aligned}
$$

Wait a minute, what do you mean to 'solve', as there are more equations than the unknowns?

1.3 Null Space

Find a basis of the null space of the following matrix A:

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & -1\\
0 & 1 & -1 & 2 & 1\\
1 & 1 & 1 & 1 & 2
\end{pmatrix}
\qquad (5)
$$

The null space of A is the set of all vector x such as $Ax=0$.

1.4 Eigenvalue Problem

Find the eigenvalues and eigenvectors of the following matrix (by hand).

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \tag{6}$$

1.5 Least-Square Models

The least-squares method minimizes the difference between observed data (P_i, Q_i) and the prediction $F(P_i)$, i.e.,

$$\text{minimize} \quad \sum_i |Q_i - F(P_i)|^2, \tag{7}$$

Work out the solution for the case of straight-line fit $F(p) = \alpha p + \beta$. That is, find expression for the fitting parameters α and β in terms of the data P_i and Q_i.

1.6 Fourier Integral

Compute by hand the Fourier integral of the function

$$f(t) = e^{-\lambda|t|}, \tag{8}$$

where $\lambda > 0$. Compare your analytic result with your FFT program result. We use the following definition for Fourier integral:

$$\tilde{f}[\omega] = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \tag{9}$$

1.7 Kepler's laws (difficult)

Based on Kepler's first law that the orbit of planet is an ellipse and the second law that the planet sweeps equal area in equal time, derive equations that predict the position of planet as a function of time [Consult many of the physics or mechanics textbooks].