CZ3102 Scientific Modeling

Tutorial/Home Work Set 2

for week 5 (starting 2 February 2004)

2.1 Least Square Normal Equation

Derive the normal equation

$$A^T A\, c = A^T Y, \tag{1}$$

for a general linear least squares problem (fitting into)

$$y_i \approx \sum_{j=1}^{k} c_j f_j(x_i), \qquad i = 1, 2, \cdots, N; \quad k < N. \tag{2}$$

We start by minimizing the sum of square errors

$$\min \sum_{i=1}^{N} \epsilon_i^2. \tag{3}$$

What are the sizes and matrix elements for the matrices A, c, and Y?

2.2 Ellipse

Using the fact that an ellipse can be drawn by fixing the sum of distances r and r', where r and r' are distance from the two focus points to a point on the elliptical curve, show that in polar coordinates the equation can be written as

$$r = \frac{a(1 - e^2)}{1 - e\,\cos\theta}, \tag{4}$$

where a is major semi-axis and e is known as eccentricity of the ellipse. The origin is at one of the focus points.

2.3 Pendulum

[Refer to page 36 (fig. 2.5) and exercise 2 on page 37] Write out the potential for a pendulum consists of a point mass m and a rigid massless rod of length l. Justify your result. The system is described by the angle the rod makes with respect to some reference direction, such as the downward direction. Find the equilibrium positions, and discuss their stability.

2.4 Supercritical Pitchfork Bifurcation

[This is exercise 4 on page 43] Graph (qualitatively) the curve

$$U_{ss}(x) = ax^2 + cx^4, \quad c > 0, \tag{5}$$

for values of a positive, zero, and negative. From these graphs, deduce the corresponding bifurcation diagram (a plot of parameter a vs. equilibrium position x_0), with solid line indicating stable equilibrium and dashed line for unstable equilibrium. This potential exemplifies the form of the *stable symmetric transition* or more picturesquely, the *supercritical pitchfork bifurcation*.

2.5 Subcritical Pitchfork Bifurcation

[This is exercise 7 on page 47] Graph the potential

$$U_{us}(x) = ax^2 + cx^4, \quad c < 0, \tag{6}$$

for values of a positive, zero, and negative. Do the same as previous problem (that is draw the bifurcation diagram). This potential exemplifies the form of the *unstable symmetric transition*, or the *subcritical pitchfork bifurcation*.

2.6 Fold Catastrophe

[Refer to page 52 and 53, fig. 2.19 and 2.20] Discuss the bifurcation and stability determined by a potential

$$V(x) = \epsilon x + cx^3, \quad c > 0, \tag{7}$$

This form of bifurcation is known as fold catastrophe. It is structurally stable — small perturbation, representing noise or imperfections, will not change the qualitative feature of the bifurcation diagram.

2.7 Wheel-Mass System

[This is exercise 10 and 11 on page 51, also see fig.2.18] (i) Suppose that the mass of the wheel is concentrated at one point. Show that the relevant potential for the system is $U(\theta) = MgR(1 - \cos\theta) - mgR\theta$. (ii) Find the equation for equilibria and show that none exist for $m > M$. (iii) How is the analysis changed if we consider asymmetric mass distributions for the wheel? (iv) Show that there is an equilibrium when $m < M$, which disappears for larger values of m.