CZ3102 Scientific Modeling

Tutorial/Home Work Set 3

for week 7 (starting 16 February 2004)

3.1 Fundamental Dimensions

[This is exercise 1 on page 65]

(i) Suppose that the fundamental dimensions are force, length, and velocity. What are the dimensions of mass and time?

(ii) What are the fundamental quantities (dimensions) of the Imperial or "English" system of units? (Hint: The mass of an object is not considered fundamental, but rather its weight is.)

(iii) What are the standards of measurement that define the Imperial system?

3.2 Physical Constants

[This is problem 1 on page 95] Suppose that the speed of light c, the gravitational constant G, and Planck's constant h are all equal to 1:

$$c = h = G = 1 \tag{1}$$

in some units. What are the units of length, mass, and time? (Give your answer in combinations of the constants as well as numerical values).

The speed of light is the length light travels in unit time; the gravitational constant G is the constant in Newton's inverse square law $F = Gm_1m_2/r^2$; and the Planck's constant relates energy of light (photo) with its frequency by $E = h\nu$.

3.3 Pendulum Period

Let us assume that the period T of oscillation of pendulum depends on the length of the pendulum l and gravitational acceleration constant g.

(a) Assume the relation

$$T = Cl^{\alpha}g^{\beta}, \tag{2}$$

determine the exponents α and β from the requirement of dimensional homogeneity. C is a dimensionless constant.

(b) For the same problem, apply the Π-procedure and Buckingham Π theorem, predict the functional relationship given T in terms of l and g.

(c) If the pendulum is a damped pendulum with additional damping parameter η (which has the dimension of inverse time), what is the new result due to Buckingham theorem?

(d) In fact, the period of a pendulum depends also on the amplitude of the oscillation. Let θ_0 be the maximum angle that the pendulum can make, in which way is the functional relationship between T and the rest of the variables modified with respect to (c)?

3.4 Water Waves

Let us use dimensional analysis to see what can we say about the hight h of water waves near beaches. The relevant quantities are density ρ, viscosity μ of water, the depth D of the water, and gravitational acceleration constant g. In addition, we think that the speed of the wind v plays an important rule. (a) Use the Buckingham Pi procedure (or by inspection) to find the independent dimensionless parameters. (b) From the result in (a), write down the functional dependence of the hight h on the rest of quantities (applying the Buckingham Pi theorem). [The viscosity is the coefficient in a share force due to fluid flow, $F = \mu dv/dy$].

3.5 Turbulence

[This is problem 18 on page 98-99.] Unless we watch a small twig or a bug, slow laminar flow presents no structure to the eye. In faster flow we see whirlpools here and there, sometimes big ones, sometimes small ones that quickly disappear. What distinguishes true turbulence is the presence of whirlpools, or eddies, of all sizes and intensities. A generally accepted concept is that whatever is powering the torrent is pumping energy into the largest eddies, from which the turbulence mechanism distributes this energy down a "cascade" of eddy scales, from large inertial scales to the smallest sizes where viscosity acts to convert the energy to heat. Kolmogorov was able to derive some information concerning the distribution of energy on the cascade, using a little dimensional analysis, along with some simplifying assumptions about the flow. The relevant parameters are: ϵ, the total rate of energy dissipation per unit mass; λ, an eddy-size variable; and the density ρ and viscosity μ of the fluid. Suppose that the energy dissipation occurs predominantly on some range of wavelengths $\lambda_{min} \leq \lambda^* < \lambda_{max}$, and is given in terms of a density on that range,

$$\epsilon = \int_{\lambda_{min}}^{\lambda_{max}} e(\lambda)\, d\lambda. \tag{3}$$

How should the density, e, depend on the flow parameters?

(a) Set

$$e \propto \mu^a \rho^b \epsilon^c \lambda^d, \tag{4}$$

and show that dimensional homogeneity requires that $b = -a$, $c = 1 - a/3$, $d = -1 - (4/3)a$.

(b) Argue, as follows, that $a = 1$. In viscous flow, where momentum is diffused, the relevant balance equation is

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \mu \nabla^2 \mathbf{v}, \tag{5}$$

where $\mathbf{v}$ is the flow velocity. But, since $\frac{1}{2}\rho v^2$ is the kinetic energy density, the dissipation rate is

$$\frac{d}{dt} \int \frac{1}{2} \rho v^2 dx = \int \rho v \frac{dv}{dt} dx. \tag{6}$$

This implies that e is proportional to μ. (Hint: The result follows if you treat the flow as one dimensional, though, of course, no real flow is.)

(c) We have, then,

$$e = C \mu \rho^{-1} \epsilon^{2/3} \lambda^{-7/3} \tag{7}$$

Argue that C is truly constant in the regime of interest and not a function of some dimensionless combination of variables.

(d) If $k = 2\pi/\lambda$ is the *wavenumber* of the eddies (essentially, the eddy density), then

$$\epsilon = (\mu/\rho) \int_{k_{min}}^{k_{max}} E(k) k^2 dk, \tag{8}$$

where $E(k) = C\epsilon^{2/3} k^{-5/3}$.