CZ3102 Scientific Modeling
Tutorial/Home Work Set 4

for week 9 (starting 1 March 2004)

4.1 Linear Differential Equations

(a) Solve the following differential equation analytically (i.e. by hand), using the standard trick of assuming $y(t) = Ae^{\lambda t}$.

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0. \tag{1}$$

(b) How many integration constants do we have? What kind of initial conditions must we use to fix the constants?

(c) Rewrite the Eq. (1) as a set of coupled first-order differential equations.

4.2 Oscillator Equation

[This is Exercise 1 on page 143]. (i) Set $k/m = \omega_0^2$, and $x_1 = \omega_0 y$, and $x_2 = \dot{y}$. Show that the oscillator equation

$$m\ddot{y} + ky = 0 \tag{2}$$

is equivalent to the system

$$\dot{x}_1 = \omega_0 x_2, \tag{3}$$
$$\dot{x}_2 = -\omega_0 x_1. \tag{4}$$

(ii) Show that in polar coordinates [of the phase space (x_1, x_2)] the equivalent system is

$$\dot{r} = 0, \tag{5}$$
$$\dot{\theta} = -\omega_0. \tag{6}$$

Find the solution of this system of differential equations. What is the result for the original variable y?

4.3 Tumour Growth Model

[This is problem 6 on page 131.] Experiments show that cell tumours grown *in vitro* have a constantly increasing diameter; the diameter growth is linear in t. Can we account for this with a simple model? Model the tumor as a sphere and suppose that growth only occurs in an outer layer of fixed thickness. [Write down a differential equation for the model].

4.4 Chemical Reaction

Assuming that the mass action law is valid, write out the differential equation systems for the concentration (number of moles per unit volume) for each of the three cases of chemical reaction. The k values above the arrows are reaction rate constants. The law of mass action says that the rate of reaction is proportional to the product of concentration C_x of source (lefthand side) molecules.

(a)

$$A + 2B \xrightarrow{k} C \tag{7}$$

(b)

$$\begin{aligned} B_2 + M &\xrightarrow{k_1} 2B' + M & (8)\\ B' + H_2 &\xrightarrow{k_2} HB + H & (9)\\ H + H &\xrightarrow{k_3} H_2 & (10) \end{aligned}$$

where B_2, B', H_2, and HB is treated as individual molecular species.

(c)

$$\begin{aligned} E + S &\xrightarrow{k_1} C, & (11)\\ C &\xrightarrow{k_2} P + E, & (12)\\ C &\xrightarrow{k_3} E + S, & (13)\\ P + E &\xrightarrow{k_4} D. & (14) \end{aligned}$$

4.5 Compartment models

[This is part of question 7 on page 132.] The so-called compartment models are very popular in physiology, especially in pharmacokinetics. There are a finite number of *compartments* acting as stores for some qualities, quantified by a mass Q_i. The rate equations are determined by mass balance:

$$\frac{d}{dt}Q_i = \sum_j R_{ij} - \sum_j R_{ji} \tag{15}$$

where R_{ij} is the mass-flow rate from compartment j to compartment i. Compartments do not have to be physical containers; in physiology, for example, they do not have to be anatomic: the quantity of a drug in all extracellular fluids could be taken as a compartment. The rate R_{ij} are typically taken to be state proportional, leading to linear equations; but if we allow more general forms it is easy to see that this is a very general modeling scheme.

(a) Start with a simple one-compartment model ($i = 1$ only) representing the blood plasma. A drug is injected intravenously and then is eliminated at a rate proportional to the amount present. What is the concentration in the blood as a function of time. A *bolus injection* is modeled by a finite amount drug given all at once at $t = 0$ (a delta function). A *short infusion* is constant for a short interval time and then is zero. A *continuous infusion* is constant for all times. What are the differences in the response of the plasma to each of these kinds of injection? Graph them.

(b) [Difficult] If drugs are administered orally, they must be absorbed through the gut. This can be modeled by allowing another compartment to represent the gut, and thus drug will flow into the plasma compartment at a rate proportional to the amount in the gut. This last rate constant depends on many factors, such as the surface area of the gut, the stomach rat flow, and blood flow. Actually, it also depends on the amount of drug present: only a certain amount can be absorbed. Write down the the resulting equations and graph the concentration of drug in the blood versus time. Show that the time needed to reach maximum concentration is independent of dose.