CZ3102 Scientific Modeling

Tutorial/Home Work Set 5

for week 11 (starting 15 March 2004)

5.1 Circuit Response and Relaxation

[This is Exercise 8 on page 108.] The voltage through an inductive element, such as a coil, is proportional to the time rate of change of current,

$$V_L = L\frac{dI}{dt}. \tag{1}$$

If a constant voltage is applied to an inductance and a resistance in series (Fig. 4.7 (middle) on page 110), what is the response of the current? What is the relaxation time? [i.e., find the differential equation for I and solve it.]

[Hint: Some basics about electric circuit. (1) In a circuit, electric current I (amount of charge per unit time flowing in the wire) is conserved. (2) For resistor, we have the Ohm's law $I = V/R$, where voltage V is the potential difference (for the electrons) between the two ends of a resistor, and R is the resistance that characterizes the resistor.]

5.2 Circuit again

[This is Exercise 9 on page 109] If a capacitance and a resistance are in parallel (Fig. 4.7 bottom), what is the response of the voltage to an applied current?

[Hint: That is, the current is fixed at I_0. Find equation for the voltage. A capacitor consists of two parallel metal plates with a small separation. For a capacitor with capacitance C, the voltage across it is $V = Q/C$. Q is the charge on the capacitor. Since electrons cannot pass through a capacitor, the flow of current causes charge acummulating on the two parallel plates (with Q and $-Q$, respectively).]

5.3 Enzyme Reaction

[This is problem 11 on page 136.] Enzymes are proteins that act as catalysts for reactions. This means that they only act in an intermediate way, and the final product finds the enzyme molecules unchanged (and looking for some more action). Enzymes are important for most biochemical reactions and are also of importance to industry, especially in food processing. A simple scheme for an enzymatic reaction is that the enzyme reacts with a species called a substrate to form a complex; a further reaction results in a desired product and free enzyme. Symbolically, this is

$$S + E \overset{k_1}{\leftrightarrow} C \overset{k_2}{\rightarrow} P + E. \tag{2}$$

The double arrow $\leftrightarrow$ denotes that the reaction is reversible, $S + E \to C$ with rate k_1 and $C \to S + E$ with rate k_{-1}.

(a) Write down a system of differential equations for the kinetics of the various concentrations.

(b) At $t = 0$, suppose that $C = P = 0$. Show that

$$E(t) + C(t) = E(0). \tag{3}$$

Eliminate E, leaving a coupled pair of equations for C and S.

5.4 System Response

[This is problem 12 on page 137, modified.] The purpose of this is to introduce some notions and terminology that are used when we think of differential equations as input-output systems. That is, the solution of the differential equations is the output to the system, which accepts as input a forcing function. The simplest case is when the equation is linear. In this scheme the differential equation

$$\dot{y} + \alpha y = f, \quad \alpha > 0, \tag{4}$$

is represented by

$$f \to \tau \to y, \tag{5}$$

where $\tau = 1/\alpha$, and this relaxation time is all that is needed to specify the system. The system could also be characterized by the solution that is obtained with the use of particular forcing functions.

(0) *Show that* the solution to the equaion is

$$y(t) = y(0)e^{-\alpha t} + e^{-\alpha t}\int_0^t e^{\alpha s} f(s)ds. \tag{6}$$

The first term on the right shows the effect of the initial conditions and is a decaying transient term. The second term is what we will called the response, and it is this that is properly the output of the system. Notice that this also may decay in time.

(a) *Impluse Response* – Suppose that $f = f_0\delta_0(t)$, where $\delta_x(t)$ is the (Dirac) delta function at $t = x$; that is, $\delta_x(t)$ has the property that

$$\int_0^\infty f(t)\delta_x(t)dt = f(x) \tag{7}$$

for any continuous function f. It can only be defined this way, via its action under the integral sign; it is not a true function. Show that the solution is

$$(y_0 + f_0)e^{-\alpha t}, \tag{8}$$

so the response solution is identical in form to the unforced equation with a different initial condition. The effect of the impulse is to simply start at a different place.

5.5 Three Coupled Pendula

Consider three pendula in a row with identical length l and mass m, each of which obeys the standard pendulum equation. Let assume that there are elastic coupling of Hooke type between the pendula, that is, the restoring force of pendulum i acting on pendulum j is

$$\Upsilon_{ij} = k(\theta_i - \theta_j). \tag{9}$$

The restoring force is only between neighbor pendula. Given the equations of the coupled system.

5.6 Chemical Reaction

[This is problem 4 on page 296.] Suppose that a molecule of a species A reacts with a molecule of B to form AB, ie., $A + B \to AB$. Let random variable $X(t)$ be the amount of A present at time t (number of A molecules). Model this with a death-type Markov process, where the transition probability is state dependent: we assume the Poisson hypotheses, where the probability of any pair of molecules reacting in some small time interval is proportional to the product of the amounts of reactants available. What are the resulting dynamics (i.e. the equations for the probability that there are n A-type molecules present)? What equation determines the mean evolution?