CZ3102 Scientific Modeling

Tutorial/Home Work Set 6

for week 13 (starting 29 March 2004)

6.1 Linear Least-Square Fits

The average number of children a Singapore woman give birth to is 1.87 in year 1990 and 1.42 in 2001, and 1.37 in 2002. If this declining trend is continuing, what is then the number of births in year 2010, assuming (a) a linear decay with time, (b) an exponential decay with time?

6.2 Stability and Bifurcation

A potential function

$$U(x) = \cos(kx) \tag{1}$$

is given. (a) Draw $U(x)$ versus x and label the stable equilibrium and unstable equilibrium points. (b) Draw the parameter k vs equilibrium x_0, solid line for stable and dash line for unstable equilibrium. Is there a bifurcation point?

6.3 Dimension Analysis

Why does a spherical soap bubble stay for a while and then burst? What factor control the size of the bubble? The relevant quantities are the radius of the bubble r, the surface tension γ of the the soap film, the pressure inside the bubble P_i and pressure outside the bubble P_o. (a) Using Buckingham Π theorem, argue what kind of relation that can exist between r, γ, P_i, P_o. (b) If time t, the duration of bubble before burst, is also considered in the dimensional analysis, what will be the new result? Is your result reasonable? Or should one consider other dynamical factors for the bubble burst such as the wind blows or the bubble touches a surface.

6.4 Ordinary Differential Equations

Solve the following differential equations analytically: (a) nonconstant coefficient, inhomogeneous

$$\frac{dy}{dt} + t\,y = e^{-t}; \tag{2}$$

(b) set of linear equations

$$\begin{aligned} \dot{x}_1 &= -x_1, & (3)\\ \dot{x}_2 &= x_2 - x_3, & (4)\\ \dot{x}_3 &= x_2 + x_3. & (5) \end{aligned}$$

6.5 Coupled ODE

In the biochemistry of blood clotting, one arrives at a pair of equations for two enzymes E_1 and E_2 that satisfy the following linear equations:

$$\dot{E}_1 = \mu_1 E_2 - k_1 E_1, \tag{6}$$
$$\dot{E}_2 = \mu_2 E_1 - k_2 E_2, \tag{7}$$

Differentiate the equation for E_1 to obtain a linear second-order scalar equation for E_1, show that we have

$$\ddot{E}_1 = -(k_1 + k_2)\dot{E}_1 - DE_1, \tag{8}$$

where $D = k_1 k_2 - \mu_1 \mu_2$.

6.6 Circuit

A resistor R, a capacitor C, and an inductor L is linked in a circular fashion as shown in Fig 5.1, page 145. A constant voltage V_0 is applied to the resistor. Write down a system of (differential) equations that completely specify the system.

6.7 Compartment Modeling

Consider the following physiological model. A patient at time $t = 0$ takes some medicine in liquid form orally. The amount of the drug is Q_0. This drug, initially all in the stomach, flows gradually and uniformly into the guts at a constant rate λ until the quantity of $Q_1(t)$ in the stomach is completely exhausted at some time t_1 later, after which there is no more flow of drug from stomach to the guts. The drug in the guts $Q_2(t)$ is absorbed into the blood stream at a rate proportional to the amount present in the guts. Let the proportionality constant be k.

(a) Determine the time t_1.

(b) Give the differential equations for the amount in the stomach Q_1 and in the guts Q_2.

(c) Solve the equations, draw qualitatively the curves $Q_1(t)$ and $Q_2(t)$, and pay particular attention to the values $t < 0$ and $t > t_1$.