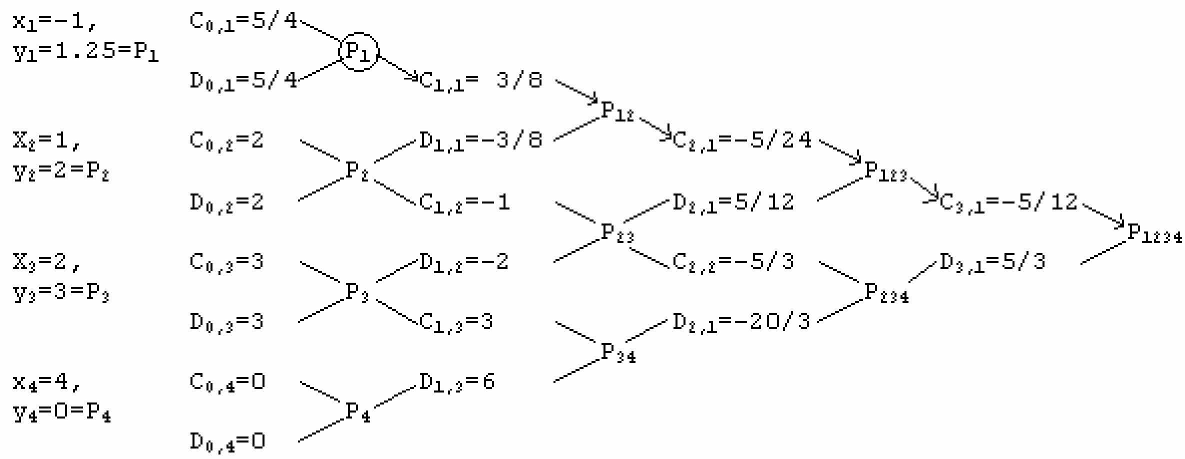
Problems for Lecture 3 (Interpolation)

1. Use Neville's algorithm to find the interpolation value at x = 0, for a cubic polynomial interpolation with 4 points (x,y) = (-1,1.25), (1,2), (2,3), (4,0). Also use the Lagrange's formula to determine the coefficients of the cubic polynomial.

Answer:

Using the small difference C & D for computations, we have:



Using $C_{m+1,i} = \dfrac{(x_i - x)(C_{m,i+1} - D_{m,i})}{x_i - x_{i+m+1}}$ and $D_{m+1,i} = \dfrac{(x_{i+m+1} - x)(C_{m,i+1} - D_{m,i})}{x_i - x_{i+m+1}}$, with x=0:

$$C_{1,1} = \frac{(x_1 - x)(C_{0,2} - D_{0,1})}{x_1 - x_2} = \frac{(-1-0)(2-5/4)}{-1-1} = \frac{3}{8}$$

$$D_{1,1} = \frac{(x_2 - x)(C_{0,2} - D_{0,1})}{x_1 - x_2} = \frac{(1-0)(2-5/4)}{-1-1} = -\frac{3}{8}$$

$$C_{1,2} = \frac{(1-0)(3-2)}{1-2} = -1\text{ , } D_{1,2} = \frac{(2-0)(-1-8)}{1-2} = -2$$

$$C_{1,3} = \frac{(2-0)(0-3)}{2-4} = 3\text{ , } D_{1,3} = \frac{(4-0)(0-3)}{2-4} = 6$$

$$C_{2,1} = \frac{(-1-0)(-1-(-3/8))}{-1-2} = -\frac{5}{24}\text{ , } D_{2,1} = \frac{(2-0)(-1-(-3/8))}{-1-2} = \frac{5}{12}$$

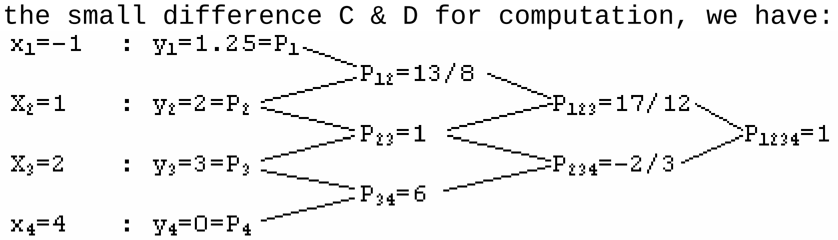
$$C_{2,2} = \frac{(1-0)(3-(-2))}{1-4} = -\frac{5}{3}\text{ , } D_{2,2} = \frac{(4-0)(3-(-2))}{1-4} = -\frac{20}{3}$$

$$C_{3,1} = \frac{(-1-0)(-5/3-5/12)}{-1-4} = -\frac{5}{12}\text{ , } D_{3,1} = \frac{(4-0)(-5/3-5/12)}{-1-4} = \frac{5}{3}$$

$$P_{1234} = C_{3,1} + C_{2,1} + C_{1,1} + P_1 = -\frac{5}{12} + \left(-\frac{5}{24}\right) + \frac{3}{8} + \frac{5}{4} = 1\text{ .}$$

$\therefore$ using Neville's algorithm, at x = 0, the interpolation value is 1.

Without using the small difference C & D for computation, we have:



At x=0,

$$P_{12}=\frac{(x-x_2)}{(x_1-x_2)}P_1+\left[1-\frac{(0-x_2)}{(x_1-x_2)}\right]P_2=\frac{(0-1)}{(-1-1)}1.25+\left[1-\frac{(0-1)}{(-1-1)}\right]2=\frac{13}{8}$$

$$P_{23}=\frac{(x-x_3)}{(x_2-x_3)}P_2+\left[1-\frac{(x-x_3)}{(x_2-x_3)}\right]P_3=\frac{(0-2)}{(1-2)}2+\left[1-\frac{(0-2)}{(1-2)}\right]3=1$$

$$P_{34}=\frac{(x-x_4)}{(x_3-x_4)}P_3+\left[1-\frac{(x-x_4)}{(x_3-x_4)}\right]P_4=\frac{(0-4)}{(2-4)}3+\left[1-\frac{(0-4)}{(2-4)}\right]0=6$$

$$P_{123}=\frac{(x-x_3)}{(x_1-x_3)}P_{12}+\left[1-\frac{(x-x_3)}{(x_1-x_3)}\right]P_{23}=\frac{(0-2)}{(-1-2)}\times\frac{13}{8}+\left[1-\frac{(0-2)}{(-1-2)}\right]1=\frac{17}{12}$$

$$P_{234}=\frac{(x-x_4)}{(x_2-x_4)}P_{23}+\left[1-\frac{(x-x_4)}{(x_2-x_4)}\right]P_{34}=\frac{(0-4)}{(1-4)}1+\left[1-\frac{(0-4)}{(1-4)}\right]6=-\frac{2}{3}$$

$$P_{1234}=\frac{(x-x_4)}{(x_1-x_4)}P_{123}+\left[1-\frac{(x-x_4)}{(x_1-x_4)}\right]P_{234}=\frac{(0-4)}{(-1-4)}\times\frac{17}{12}+\left[1-\frac{(0-4)}{(-1-4)}\right]\times\frac{-2}{3}=1$$

∴ using Neville's algorithm, at x = 0, the interpolation value is 1.

Next, we use the Lagrange's formula:

$$l_1(x)=\frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}=\frac{(x-1)(x-2)(x-4)}{(-1-1)(-1-2)(-1-4)}=\frac{1}{-30}\left(x^3-7x^2+14x-8\right)$$

$$l_2(x)=\frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}=\frac{(x+1)(x-2)(x-4)}{(1+1)(1-2)(1-4)}=\frac{1}{6}\left(x^3-5x^2+2x+8\right)$$

$$l_3(x)=\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}=\frac{(x+1)(x-1)(x-4)}{(2+1)(2-1)(2-4)}=\frac{1}{-6}\left(x^3-4x^2-x+4\right)$$

$$l_4(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}=\frac{(x+1)(x-1)(x-2)}{(4+1)(4-1)(4-2)}=\frac{1}{30}\left(x^3-2x^2-x+2\right)$$

For verification, $\sum_{k=1}^{4} l_k(x)=1$.

Hence, $f(x)=\sum_{k=1}^{4} l_k(x)y_k=\frac{1.25}{-30}\left(x^3-7x^2+14x-8\right)+\frac{2}{6}\left(x^3-5x^2+2x+8\right)-\frac{3}{6}\left(x^3-4x^2-x+4\right)$

$$f(x)=\left(\frac{1\times1.25}{-30}+\frac{1\times2}{6}+\frac{1\times3}{-6}\right)x^3+\left(\frac{7\times1.25}{30}+\frac{-5\times2}{6}+\frac{4\times3}{6}\right)x^2+\left(\frac{14\times1.25}{-30}+\frac{2\times2}{6}+\frac{1\times3}{6}\right)x$$
$$+\left(\frac{8\times1.25}{30}+\frac{8\times2}{6}+\frac{4\times3}{-6}\right)=-\frac{5}{24}x^3+\frac{5}{8}x^2+\frac{7}{12}x+1$$

2. Given the same 4 points as above, determine the cubic splines with nature boundary condition (second derivatives equal to 0 at the boundary). Give the cubic polynomials in each of the three intervals. Use the method outlined in NR page 113 to 116, or direct fitting to cubic polynomials with the proper conditions, and show that the results are the same by two different approaches.

Answer:

From Page 114,

$y=Ay_j+By_{j+1}+Cy''_j+Dy''_{j+1}$ - Equation (3.3.3)

where $A\equiv\frac{x_{j+1}-x}{x_{j+1}-x_j}, B\equiv\frac{x-x_j}{x_{j+1}-x_j}$ - Equation (3.3.2) and

$C\equiv\frac{1}{6}\left(A^3-A\right)\left(x_{j+1}-x_j\right)^2, D\equiv\frac{1}{6}\left(B^3-B\right)\left(x_{j+1}-x_j\right)^2$ - Equation (3.3.4)

Furthermore, from page 115,

$$\frac{x_j - x_{j-1}}{6} y''_{j-1} + \frac{x_{j+1} - x_{j-1}}{3} y''_j + \frac{x_{j+1} - x_j}{6} y''_{j+1} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$ - Equation (3.3.7)

As we are looking for natural cubic spline, $y''_1 = y''_4 = 0$.
Thus, for j=2, equation 3.3.7 becomes:

$$\frac{1-(-1)}{6}(0) + \frac{2-(-1)}{3} y''_2 + \frac{2-1}{6} y''_3 = \frac{3-2}{2-1} - \frac{2-5/4}{1-(-1)}$$

$$y''_2 + \frac{1}{6} y''_3 = \frac{5}{8}$$ - Equation (2a-1)

For j=3, equation 3.3.7 becomes:

$$\frac{2-1}{6} y''_2 + \frac{4-1}{3} y''_3 + \frac{4-2}{6}(0) = \frac{0-3}{4-2} - \frac{3-2}{2-1}$$

$$\frac{1}{6} y''_2 + y''_3 = -\frac{5}{2}$$ - Equation (2a-2)

Solving for y''_2, y''_3:

Eqn(2a-1) – 6 × Eqn(2a-2): $\frac{1}{6} y''_3 - 6 y''_3 = \frac{5}{8} + \frac{30}{2} \Rightarrow y''_3 = \frac{125}{8} \times \frac{-6}{35} = -\frac{75}{28}$

Thus, Eqn (2a-1): $y''_2 + \frac{1}{6}\left(-\frac{75}{28}\right) = \frac{5}{8} \Rightarrow y''_2 = \frac{5}{8} + \frac{25}{56} = \frac{15}{14}$

Using Eqn (3.3.3), (3.3.2), and (3.3.4) with the solved values for y''_2, y''_3, the cubic polynomial P_1 for the interval (x_1, x_2) i.e., (-1,1), is given by:

$$P_1 = y = A_1 y_1 + B_1 y_2 + C_1 y''_1 + D_1 y''_2, \text{ where}$$

$$A_1 = \frac{x_2 - x}{x_2 - x_1} = \frac{1-x}{1-(-1)} = \frac{1}{2}(1-x), B_1 = \frac{x - x_1}{x_2 - x_1} = \frac{x-(-1)}{1-(-1)} = \frac{1}{2}(1+x)$$

C_1 needs not be evaluated as $y''_1 = 0$.

$$D_1 = \frac{1}{6}\left(B_1^3 - B_1\right)(x_2 - x_1)^2 = \frac{1}{6}\left(\frac{1}{2}(1+x)\right)\left[\left(\frac{1}{2}(1+x)\right)^2 - 1\right](1-(-1))^2$$

$$= \frac{1}{12}(1+x)\left[(1+x)^2 - 4\right] = \frac{1}{12}(1+x)\left[(1+x-2)(1+x+2)\right] = \frac{1}{12}(1+x)(x-1)(x+3)$$

Hence, $P_1 = \frac{1}{2}(1-x)\left(\frac{5}{4}\right) + \frac{1}{2}(1+x)(2) + C_1(0) + \frac{1}{12}(1+x)(x-1)(x+3)\left(\frac{15}{14}\right)$

$$= \frac{5}{8}(1-x) + (1+x) + \frac{15}{168}\left(x^3 + 3x^2 - x - 3\right) = \frac{5}{56}x^3 + \frac{15}{56}x^2 + \frac{2}{7}x + \frac{19}{14}$$

The cubic polynomial P_2 for the interval (x_2, x_3) i.e., (1,2), is given by:

$$P_2 = y = A_2 y_2 + B_2 y_3 + C_2 y''_2 + D_2 y''_3,$$

where $A_2 = \frac{x_3 - x}{x_3 - x_2} = \frac{2-x}{2-1} = (2-x), B_2 = \frac{x - x_2}{x_3 - x_2} = \frac{x-1}{2-1} = (x-1)$

$$C_2 = \frac{1}{6}\left(A_2^3 - A_2\right)(x_3 - x_2)^2 = \frac{1}{6}(2-x)\left[(2-x)^2 - 1\right](2-1)^2$$

$$= \frac{1}{6}(2-x)\left[(2-x)^2 - 1\right] = \frac{1}{6}(2-x)\left[(2-x-1)(2-x+1)\right] = \frac{1}{6}(2-x)(x-1)(x-3)$$

$$D_2 = \frac{1}{6}\left(B_2^3 - B_2\right)\left(x_3 - x_2\right)^2 = \frac{1}{6}(x-1)\left[(x-1)^2 - 1\right](2-1)^2$$

$$= \frac{1}{6}(x-1)\left[(x-1)^2 - 1\right] = \frac{1}{6}(x-1)\left[(x-1-1)(x-1+1)\right] = \frac{1}{6}x(x-1)(x-2)$$

Hence, $P_2 = (2-x)(2) + (x-1)(3) + \frac{1}{6}(2-x)(x-1)(x-3)\left(\frac{15}{14}\right) + \frac{1}{6}x(x-1)(x-2)\left(-\frac{75}{28}\right)$

$$= 4 - 2x + 3x - 3 + \frac{5}{28}\left(-x^3 + 6x^2 - 11x + 6\right) - \frac{25}{56}\left(x^3 - 3x^2 + 2x\right) = -\frac{5}{8}x^3 + \frac{135}{56}x^2 - \frac{13}{7}x + \frac{29}{14}$$

Finally, the cubic polynomial P_3 for the interval (x_3, x_4) i.e., (2,4), is given by:

$P_3 = y = A_3 y_3 + B_3 y_4 + C_3 y_3'' + D_3 y_4''$, where $A_3 = \frac{x_4 - x}{x_4 - x_3} = \frac{4-x}{4-2} = \frac{1}{2}(4-x)$, B_3 needs not be evaluated as $y_4 = 0$

and D_3 needs not to be evaluated.

$$C_3 = \frac{1}{6}\left(A_3^3 - A_3\right)\left(x_4 - x_3\right)^2 = \frac{1}{6}\left(\frac{1}{2}(4-x)\right)\left[\left(\frac{1}{2}(4-x)\right)^2 - 1\right](4-2)^2$$

$$= \frac{1}{12}(4-x)\left[(4-x)^2 - 4\right] = \frac{1}{12}(4-x)\left[(4-x-2)(4-x+2)\right] = \frac{1}{12}(4-x)(2-x)(6-x)$$

D_3 needs not be evaluated as $y_4'' = 0$.

Hence, $P_3 = \frac{1}{2}(4-x)(3) + B_3(0) + \frac{1}{12}(4-x)(2-x)(6-x)\left(-\frac{75}{28}\right) + D_3(0)$

$$= 6 - \frac{3}{2}x - \frac{25}{112}\left(-x^3 + 12x^2 - 44x + 48\right) = \frac{25}{112}x^3 - \frac{75}{28}x^2 + \frac{233}{28}x - \frac{33}{7}$$

Next, we use direct fitting to cubic polynomials with nature boundary condition.

We first define the cubic polynomial for the interval (x_i, x_{i+1}) to be:

$$P_i(x) = A_i x^3 + B_i x^2 + C_i x + D_i$$

Thus, its first and second derivatives are given by:

$$P_i' = 3A_i x^2 + 2B_i x + C_i \quad \text{and} \quad P_i'' = 6A_i x + 2B_i$$

Given that x_1=-1, we have:

$P_1(x_1) = A_1 x_1^3 + B_1 x_1^2 + C_1 x_1 + D_1 = y_1 \Rightarrow -A_1 + B_1 - C_1 + D_1 = \frac{5}{4}$ - Equation (2b-1)

Next, consider at x_2=1, we have:

$P_1(x_2) = A_1 x_2^3 + B_1 x_2^2 + C_1 x_2 + D_1 = y_2 \Rightarrow A_1 + B_1 + C_1 + D_1 = 2$ - Equation (2b-2)

$P_2(x_2) = A_2 x_2^3 + B_2 x_2^2 + C_2 x_2 + D_2 = y_2 \Rightarrow A_2 + B_2 + C_2 + D_2 = 2$ - Equation (2b-3)

Next, consider at x_3=2, we have:

$P_2(x_3) = A_2 x_3^3 + B_2 x_3^2 + C_2 x_3 + D_2 = y_3 \Rightarrow 8A_2 + 4B_2 + 2C_2 + D_2 = 3$ - Equation (2b-4)

$P_3(x_3) = A_3 x_3^3 + B_3 x_3^2 + C_3 x_3 + D_3 = y_3 \Rightarrow 8A_3 + 4B_3 + 2C_3 + D_3 = 3$ - Equation (2b-5)

Finally, consider at at x_4=4, we have:

$P_3(x_4) = A_3 x_4^3 + B_3 x_4^2 + C_3 x_4 + D_3 = y_4 \Rightarrow 64A_3 + 16B_3 + 4C_3 + D_3 = 0$ - Equation (2b-6)

We have 12 unkowns (A_i, B_i, C_i, D_i, where i=1,2,3) but only 6 equations. We now make use of the conditions that the 1st derivatives have to be smooth and the 2nd derivatives must be continuous within an interval and at its boundaries.

$$P_1'(x_2) = P_2'(x_2), \; \therefore 3A_1 + 2B_1 + C_1 = 3A_2 + 2B_2 + C_2$$ - Equation (2b-7)

$$P_1''(x_2) = P_2''(x_2), \; \therefore 6A_1 + 2B_1 = 6A_2 + 2B_2$$ - Equation (2b-8)

$$P_2'(x_3) = P_3'(x_3), \; \therefore 12A_2 + 4B_2 + C_2 = 12A_3 + 4B_3 + C_3$$ - Equation (2b-9)

$$P_2''(x_3) = P_3''(x_3), \; \therefore 12A_2 + 2B_2 = 12A_3 + 2B_3$$ - Equation (2b-10)

Finally, for natural cubic spline, we have:

$$P_1''(x_1) = 0, \; \therefore -6A_1 + 2B_1 = 0$$ - Equation (2b-11)

$$P_3''(x_4) = 0, \; \therefore 24A_3 + 2B_3 = 0$$ - Equation (2b-12)

We now have 12 equations for 12 unknowns. Solving using the NR ludcmp () and lubksb() via computer, we have the following answers:

$$P_1(x) = 0.0893x^3 + 0.2679x^2 + 0.2857x + 1.357$$
$$P_2(x) = -0.625x^3 + 2.411x^2 - 1.857x + 2.071$$
$$P_1(x) = 0.2332x^3 - 2.679x^2 + 8.321x - 4.714$$

Comparing the coefficients obtained by the two methods, it is observed that both approaches give the same results.

i	**A_i**		**B_i**		**C_i**		**D_i**	
	NR	**Direct**	**NR**	**Direct**	**NR**	**Direct**	**NR**	**Direct**
1	15/168≈ 0.0893	0.0893	15/56≈ 0.2679	0.2679	2/7≈ 0.2857	0.2857	19/14≈ 1.357	1.357
2	-5/8= -0.625	-0.625	135/56≈ 2.411	2.411	-13/7≈ 1.857	-1.857	29/14≈ 2.071	2.071
3	25/112≈ 0.2232	0.2332	-75/28≈ -2.679	-2.679	233/28≈ 8.321	8.321	-33/7≈ -4.714	-4.714