PC5215 Numerical Recipes with Applications

Midterm test, 3 October 2024, 10:00 – 11:45

1. We consider pieces of Python codes. Given the input as shown, write the output when the code is run:

    a.
    ```
    for i in range(4):
      print(i)
    ```

    b.
    ```
    lis = [1, 2, 3]
    lis.pop()
    ```

    c.
    ```
    x=[ ]
    if x:
      print(1)
    else:
      print(2)
    ```

    d.
    ```
    def func(y):
       y[0] += 1
    y=[0]
    func(y)
    y
    ```

    e. 5 ^ 2

    f. {“A”, ”B”, ”C”} | {1,2}

    g. 2//3, 4%3

    h. What does the bit pattern 3F80 0000 (in hex notation) denote for the IEEE 32-bit float number?

Ans: a) 0

1

2

3

b) 3 c) 2 d) [1] e) 7, f) { 1, 2, “A”, “B”, “C””}, g) (0, 1) h) 1.0

2. Neville's algorithm is a way to compute the interpolation value at x by the polynormal $f(x)$, but we can also use it to construct the polynomial itself. Given the 4 points of (x_i, y_i) as, (1,0), (2,6), (3,6), (4,12), determine the polynomial via Neville's algorithm by hand. It is sufficient to construct the P tableau as your results.

   The basic formula to use is
   $\lambda = \frac{x - x_{i+m}}{x_i - x_{i+m}}$, $P_{12} = \lambda P_1 + (1 - \lambda) P_2$

   $\lambda_{12} = 2 - x, \quad \lambda_{23} = 3 - x, \qquad \lambda_{34} = 4 - x$
   $P_{12} = 6x - 6,\ P_{23} = 6\,, P_{34} = 6x - 12,$
   $\lambda_{13} = \dfrac{3 - x}{2}, \qquad \lambda_{24} = (4 - x)/2$
   $P_{123} = -3x^2 + 15x - 12, \qquad P_{234} = 3x^2 - 15\,x + 24$
   $\lambda_{14} = (4 - x)/3$
   $P_{1234} = 2x^3 - 15x^2 + 37x - 24.$

3. (a) State the main idea of Gaussian quadrature and the theorem concerning the accuracy of Gaussian quadrature in terms of integration of polynomials. (b) Derive an integration formula for Gaussian quadrature with two points, $\int_0^h f(x)dx = w_1 f_1 + w_2 f_2$, i.e., determine the weights w_1 and w_2, and x_1 and x_2, and state the accuracy in terms of h.

   (a) *The idea is to displace the x values arbitrarily instead of using equally spaced abscissas. The advantage is that by doing this, all polynomials can be integrated exactly up to order 2N -1 for an N-point formula. If we use two-point formula, then all, 1, x, x^2, x^3 will be integrated exactly.*
   (b) *$P_0 = 1$, $P_1 = x + b$, using $\int P_1 dx = 0$, we find b = -h/2. $P_2 = x^2 + cx + d$. Using $\int P_2 dx = 0$ and $\int P_1\ P_2\ dx = 0$, we find $P_2 = x^2 - hx + h^2/6$. So the value x_i should be the root of $P_2(x) = 0$, which gives $\frac{h}{2}(1 \pm \sqrt{1/3})$. To determine the weight w, we ask for P_0 and P_1 to be integrated exactly, which gives $\int P_0 dx = w_1 + w_2 = h$, and $\int P_1 dx = w_1\left(x_1 - \frac{h}{2}\right) + w_2\left(x_2 - \frac{h}{2}\right) = 0$, or $w_1 = w_2 = \frac{h}{2}$. So the integration formula is*

$$\int_0^h f(x)dx = \frac{h}{2}[f(x_1) + f(x_2)]$$

   where $x_1 = \frac{h}{2}(1 + \sqrt{\frac{1}{3}})$ $x_2 = \frac{h}{2}(1 - \sqrt{\frac{1}{3}})$. The errors is $O(h^5)$ because error occurred when we integrate x^4.

4. This is about random numbers. (a) State the linear congruential method for uniformly distributed random integers and discuss how this can be implemented as a computer code. (b) Suppose we have three uncorrelated and independent, uniformly distributed real numbers in [0,1), call them ξ_1, ξ_2, and ξ_3. What is the probability density $P(x)$ of the random variable $x = \xi_1 + \xi_2 + \xi_3$?

(a) Using integers by $x_{n+1} = (a x_n + c) \bmod m$. Here all quantities are integers. On a computer, it is most efficient if m is the word size (e.g. 64-bit word), then the mod operation is not necessary, and the integers should be unsigned.

(b) Define the cumulative distribution $F(x) = P(\xi_1 + \xi_2 + \xi_3 \le x) = \iiint_{\xi_1+\xi_2+\xi_3<x} d\xi_1 d\xi_2 d\xi_3$. We can simplify by considered $y = \xi_1 + \xi_2$. The distribution of P(y) = y if 0 < y < 1, and 2-y if 1 < y < 2. Then the distribution of $x = \xi_3 + y$ can be computed as $F(x) = \iint_{y+\xi_3<x} d\xi_3 dy\, P(y)$. This integral need to be done in three cases, 0< x < 1, 1 < x < 2, and 2 < x < 3. The result for P(x)=F'(x) is

$$P(x) = \begin{cases} \frac{1}{2}x^2, \ 0 < x < 1 \\ -x^2 + 3x - \frac{3}{2}, \ 1 < x < 2 \\ \frac{1}{2}(3-x)^2, \ 2 < x < 3 \end{cases}$$

The end