PC5215, Numerical Recipes with Applications

Midterm Test, 2 October 2009

1. A bit pattern of the IEEE 754 single precision floating point number (float in C) is given as `0100 0001 0010 0100 0000 0000 0000 0000`. What real decimal value does it represent? Note that the highest bit is for the sign, the next 8 bits are for the biased exponent, the rest 23 bits are for the fractional part.

   Sign bit is 0 for a positive number; the biased exponent is e=100 0001 0 = 128 + 2 = 130; the fractional part is 0.010 01 = ¼ + 1/(32) = 0.28125. The floating number is $2^{e-127} \times (1 + 0.28125) = 10.25$.

2. Consider the special polynomials $l_i(x)$ used in the Lagrange interpolation formula. (a) If the number of interpolation points $N = 4$ at x_1 to x_4 equals −1, 0, 2, 4, respectively, give the first polynomial $l_1(x)$. (b) Show that $\sum_{j=1}^{N} l_j(x) = 1$ for any N.

   (a) $l_1(x) = \frac{x(x-2)(x-4)}{(-1-0)(-1-2)(-1-4)} = -\frac{1}{15}x(x-2)(x-4)$. *(b) Consider a function* $f(x) \equiv 1$. *Using the interpolation formula for* $f(x) = \sum_{j=1}^{N} y_i l_j(x) = 1$, *we find that the only choice for y is 1. Because the required interpolation result is unique, we must have the result for any x and N.*

3. Consider two-point gaussian quadrature formula of the form $\int_0^h f(x)dx = w_1 f(x_1) + w_2 f(x_2)$. (a) Which of the following polynomials can be integrated exactly: 1, x, x^2, x^3, x^4, x^5? Give your intuition as why this is so. (b) Determine the abscissas x_1 and x_2 and weights w_1 and w_2.

   (a) for polynomials 1, x, x^2, x^3, *the integration formula is exact. This is because we have 4 degrees of freedom to fix the formula. (b) Let* $P_0 = 1$, $P_1 = x + c$, *by orthogonality of* P_0 *and* P_1, *we find* $\int_0^h (x+c)dx = \frac{1}{2}h^2 + ch = 0$. *So* $c = -h/2$.

   Similarly, assuming $P_2 = x^2 + bx + c$, *we determine b and c by* $\int_0^h P_0 P_2 dx = 0$, *and*

$\int_0^h P_1 P_2 dx = 0$, which gives b = -h, c= $h^2/6$. The roots of P_2 are $x_{1,2}/h = \frac{1}{2} \pm \sqrt{\frac{1}{12}}$.
To determine the weights, we use that fact that the formula is exact for polynomials 1, and x. This gives $h = w_1 + w_2$, $h^2/2 = w_1x_1 + w_2x_2$. The solution is $w_1=w_2=h/2$.

4. We are interested to generate a two-dimensional distribution with probability density of the form $e^{-|x|-y^2}\frac{1}{1+|xy|^2}$ for variables $-\infty < x, y < +\infty$. We want to use the Metropolis Monte Carlo algorithm to sample such a distribution. Write a set of pseudo-code to do so (not really C program but a set of outline steps that can be easily transcribed into a program).

```
Set x =0, y=0, a = 0.1;
M is a number larger than typical equilibration time.
for(i=0; i < M; ++i)  {
        ξ1 = drand48();
        ξ2 = drand64();
        x1 = x + (2ξ1-1)a;
        y1 = y + (2ξ2-1)a;
        f1 = exp(-fabs(x1)-y1*y1)/(1+fabs(x1*y1)^2);
        f = exp(-fabs(x)-y*y)/(1+fabs(x*y)^2);
        r = f1/f;
        if( drand48() < r) {
                x = x1;
                y = y1;
        }
}
return (x,y)
```