NATIONAL UNIVERSITY OF SINGAPORE

PC2135 Thermodynamics and Statistical Mechanics

(Semester II: AY 2023-24)

Time Allowed: 2 Hours

INSTRUCTIONS TO STUDENTS

1. Please write only your student number. Do not write your name.
2. This assessment paper contains 4 questions. It comprises 3 printed pages.
3. Students are required to answer ALL questions. The questions carry equal marks.
4. The answers are to be written in the answer book.
5. Students should write the answers for each question on a new page.
6. This is a CLOSED BOOK examination.

1. This problem considers the thermodynamic properties of black holes.
    a. Although we don't know exactly what they are, black holes have sizes. The black hole radius R can be estimated based on the fact that even light cannot escape from them. Assuming the mass of a black hole is M, determine the escape velocity, and from it, determine the black hole radius.
    b. The entropy of the black hole can be estimated by the fact that each particle can have entropy of order the Boltzmann constant, and all the mass M are from the energy of massless photons. Using the result of part a, perform such an estimate to entropy S. The expression involves the Boltzmann constant k, Planck constant h, speed of light c, gravitational constant G, and mass of the black hole M.
    c. A nonzero entropy implies a temperature. Estimate the temperature T of a black hole.

2. We consider the relationship between the Gibbs free energy with the enthalpy in thermodynamics.
    a. Based on the definition of Gibbs free energy and the thermodynamic identity, show that the Gibbs free energy satisfies a differential equation

    $$G = H + T\left(\frac{\partial G}{\partial T}\right)_{P,N}.$$

    Here, G is Gibbs free energy, H is enthalpy, T is temperature, and P is pressure.
    b. Solve this first-order differential equation, i.e., express G in terms of an integral involving H, considering the enthalpy as a known function of temperature.
    c. Derive the Gibbs-Helmholtz equation which is

    $$\left(\frac{\partial(G/T)}{\partial T}\right)_{P,N} = -\frac{H}{T^2}.$$

3. Consider one free classical particle in a box of volume $V = L^3$. The energy of the particle is $\frac{1}{2}m\mathbf{v}^2$.
    a. Calculate the partition function z of one particle in volume V by performing an integral in a six-dimensional phase space $(\mathbf{r}, \mathbf{p})$.
    b. If there are N non-interacting particles in the same volume, i.e., an ideal gas, what is the partition function Z in terms of the one-particle partition function z? Determine the pressure from the partition function.
    c. If the particle in part a is now in a gravitational field with a potential energy mgH, assuming H is much larger than the size of the box L, what is the partition function Z? Determine the chemical potential μ as a function of the height H.

You may need the Gauss integral $\int_{-\infty}^{+\infty} e^{-x^2/2} dx = \sqrt{2\pi}$.

4. Consider an electron confined in a one-dimensional space in the interval $(0, L)$ as the particle-in-a-box problem.
    a. What are the energy levels of a single electron?
    b. What are the energy levels of the system if we have two electrons, ignoring the electron Coulomb interaction and spin degeneracy but taking into account the Pauli exclusion principle? Write down the expression for the partition function.
    c. It is unlikely the partition function in b can be calculated exactly if we have a fixed number of fermionic electrons. However, if we fixed the chemical potential μ instead of the number of electrons, the partition function (known as the grand partition function) for a collection of free electrons can be expressed exactly. Give the corresponding expression for this case.

--- End of Paper --- [WJS]