PC2230 Thermodynamics and Statistical Mechanics, homework 3

Due 2 Mar Thursday 11:59PM.

Problem 3.25 (60 marks), problems 3.13, 3.36, 3.37, 4.1 (10 marks each) [total 100 marks].

Problem 3.25. In Problem 2.18 you showed that the multiplicity of an Einstein solid containing N oscillators and q energy units is approximately

$$\Omega(N,q) \approx \left(\frac{q+N}{q}\right)^q \left(\frac{q+N}{N}\right)^N.$$

(a) Starting with this formula, find an expression for the entropy of an Einstein solid as a function of N and q. Explain why the factors omitted from the formula have no effect on the entropy, when N and q are large.

(b) Use the result of part (a) to calculate the temperature of an Einstein solid as a function of its energy. (The energy is $U = q\epsilon$, where ϵ is a constant.) Be sure to simplify your result as much as possible.

(c) Invert the relation you found in part (b) to find the energy as a function of temperature, then differentiate to find a formula for the heat capacity.

(d) Show that, in the limit $T \to \infty$, the heat capacity is $C = Nk$. (Hint: When x is very small, $e^x \approx 1 + x$.) Is this the result you would expect? Explain.

(e) Make a graph (possibly using a computer) of the result of part (c). To avoid awkward numerical factors, plot C/Nk vs. the dimensionless variable $t = kT/\epsilon$, for t in the range from 0 to about 2. Discuss your prediction for the heat capacity at low temperature, comparing to the data for lead, aluminum, and diamond shown in Figure 1.14. Estimate the value of ϵ, in electron-volts, for each of those real solids.

(f) Derive a more accurate approximation for the heat capacity at high temperatures, by keeping terms through x^3 in the expansions of the exponentials and then carefully expanding the denominator and multiplying everything out. Throw away terms that will be smaller than $(\epsilon/kT)^2$ in the final answer. When the smoke clears, you should find $C = Nk[1 - \frac{1}{12}(\epsilon/kT)^2]$.

Problem 3.13. When the sun is high in the sky, it delivers approximately 1000 watts of power to each square meter of earth's surface. The temperature of the surface of the sun is about 6000 K, while that of the earth is about 300 K.

(a) Estimate the entropy created in one year by the flow of solar heat onto a square meter of the earth.

(b) Suppose you plant grass on this square meter of earth. Some people might argue that the growth of the grass (or of any other living thing) violates the second law of thermodynamics, because disorderly nutrients are converted into an orderly life form. How would you respond?

Problem 3.36. Consider an Einstein solid for which both N and q are much greater than 1. Think of each oscillator as a separate "particle."

(a) Show that the chemical potential is

$$\mu = -kT \ln\left(\frac{N+q}{N}\right).$$

(b) Discuss this result in the limits $N \gg q$ and $N \ll q$, concentrating on the question of how much S increases when another particle carrying no energy is added to the system. Does the formula make intuitive sense?

Problem 3.37. Consider a monatomic ideal gas that lives at a height z above sea level, so each molecule has potential energy mgz in addition to its kinetic energy.

(a) Show that the chemical potential is the same as if the gas were at sea level, plus an additional term mgz:

$$\mu(z) = -kT \ln\left[\frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\right] + mgz.$$

(You can derive this result from either the definition $\mu = -T(\partial S/\partial N)_{U,V}$ or the formula $\mu = (\partial U/\partial N)_{S,V}$.)

(b) Suppose you have two chunks of helium gas, one at sea level and one at height z, each having the same temperature and volume. Assuming that they are in diffusive equilibrium, show that the number of molecules in the higher chunk is

$$N(z) = N(0)e^{-mgz/kT},$$

in agreement with the result of Problem 1.16.

Problem 4.1. Recall Problem 1.34, which concerned an ideal diatomic gas taken around a rectangular cycle on a PV diagram. Suppose now that this system is used as a heat engine, to convert the heat added into mechanical work.

(a) Evaluate the efficiency of this engine for the case $V_2 = 3V_1$, $P_2 = 2P_1$.

(b) Calculate the efficiency of an "ideal" engine operating between the same temperature extremes.