NATIONAL UNIVERSITY OF SINGAPORE

PC5202 Advanced Statistical Mechanics

(Semester II: AY 2008-09, 29 April 09)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This is an OPEN BOOK examination.
2. This examination paper contains 5 questions and comprises 3 printed pages.
3. Answer ALL the questions.
4. Answers to the questions are to be written in the answer books.
5. Each question carries 20 marks.

1. Consider a quantum ideal gas of fermionic particles of spin ½ (e.g. electrons) in one dimension. The particles are confined in space from 0 to L with a periodic boundary condition. The energy of a single particle is

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m}, \quad k = \frac{2\pi l}{L}, \ l = \cdots, -2, -1, 0, 1, 2, \cdots .$$

   a. Derive the formula for the density of states $D(\varepsilon)$, counting the spin degeneracy.
   b. Express the grand partition function Ξ in terms of an integral involving $D(\varepsilon)$ and chemical potential μ.
   c. Compute the heat capacity C of the system of N particles in the low temperature limit.

a. The spacing in wave vector k is 2π/L, so the density in k-space is L dk/(2π). Map energy ε to k, $k = \sqrt{2m\varepsilon/\hbar^2}$ *,multiply by 4 (2 for spin, another 2 for + and – k for each ε), we get* $D(\varepsilon) = \frac{L}{\pi\hbar}\sqrt{\frac{2m}{\varepsilon}}$.

b. $\ln\Xi = \int_0^{+\infty} d\varepsilon D(\varepsilon)\ln(1+e^{-\beta(\varepsilon-\mu)})$.

c. Let ε_F *be the Fermi energy (μ at T = 0), then because variation occurs only near Fermi energy:*

$$C = \frac{dU}{dT} = \int_0^{+\infty} d\varepsilon(\varepsilon - \varepsilon_F)D(\varepsilon)\frac{\partial f}{\partial T}$$

$$\approx D(\varepsilon_F)\int_0^{+\infty} d\varepsilon(\varepsilon - \varepsilon_F)\frac{\partial f}{\partial T}$$

$$\approx D(\varepsilon_F)k_B^2 T\int_{-\infty}^{+\infty}\frac{x^2 e^x}{(e^x+1)^2}dx = D(\varepsilon_F)k_B^2 T\frac{\pi^2}{3}$$

2. Consider a model equation of state of a magnetic system which, near the critical point $t=(T-T_c)/T_c= 0$, can be written as

   $$H \approx aM(t+bM^2)^{7/4},$$

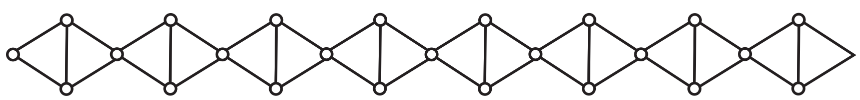
   where H is the magnetic field, and M is the total magnetization, a and b are positive constants.

   a. Determine the magnetic field dependence of the Helmholtz free energy, $F(H,t)$, at the critical temperature $t = 0$.
   b. Find the critical exponent β and γ defined by the spontaneous magnetization $M \propto t^{\beta}$ and susceptibility at zero magnetic field $\chi \propto t^{-\gamma}$.

   a. *$dF=-S\,dT-M\,dH$, $t=0$, $dT=0$, so $dF = -M\,dH$, and $H=ab^{7/4}M^{9/2}$, integrate we get $F \propto const - H^{11/9}$.*
   b. *Set $H=0$, we get $M=0$, or $M = \pm\sqrt{-t/b}$, so $\beta=1/2$. consider $t>0$, $M=0$, we find $dH = a\,t^{7/4}dM$, so $\gamma=7/4$.*

3. Consider the following quasi-one-dimensional chain of Ising model formed by left pointing and right pointing triangles. The energy of the system is the sum of the nearest neighbor interactions (indicated by the lines connecting points in the figure below) of the form $-J\sigma_i\sigma_j$ without magnetic field, where J is the exchange coupling constant.

   a. Based on general understanding of phase transitions in the Ising systems, will the quasi-one-dimensional chain below show a phase transition at a finite temperature $T_c > 0$?
   b. Give the matrix elements of a transfer matrix P, such that the partition function is $Z = \mathrm{Tr}\, P^N$ (no need to solve the eigenvalue problem of P).



   a. *No phase transitions in 1D with short-range interactions at finite temperature.*
   b. *Sum over the spins on the two sites of vertical line, the transfer matrix is*

   $$\begin{pmatrix} e^{5K}+e^{-3K}+2e^{-K} & 2e^{K}+2e^{-K} \\ 2e^{K}+2e^{-K} & e^{5K}+e^{-3K}+2e^{-K} \end{pmatrix}$$

4. Consider a one-dimensional nearest neighbor Ising model defined on a ring with sites labeled from 1 to N, where the first site 1 is connected to the last site N due to periodic boundary condition. The energy is given in the standard form

$$H(\sigma) = -J\sum_{i=1}^{N} \sigma_i \sigma_{(i+1)\,\mathrm{mod}\,N} \ .$$

   a. Give the two nonzero terms of the partition function in a high-temperature expansion in the variable $x = \tanh[J/(k_B T)]$.
   b. Compute the pair correlation function, $g(j) = \left\langle \sigma_1 \sigma_{(1+j)\,\mathrm{mod}\,N} \right\rangle$, based on high-temperature expansion in part (a). Pay attention to the effect of finite N and periodic boundary condition.

a. $Z = 2^N \cosh^N(K)\,(1 + x^N)$

b. $g(j) = \dfrac{x^j + x^{N-j}}{1 + x^N}$. *Note that there are two ways for a nonzero contribution, connect from site 1,2 3, …1 + j, or from 1+j, 2+j, …, N-1, N, 1.*

5. The Langevin equation of a particle of unit mass under a constant driven force is

$$\frac{dv}{dt} = -\gamma v + f + R(t),$$

where f is a constant and the random white noise satisfies

$$\langle R(t) \rangle = 0, \quad \langle R(t)R(t') \rangle = C\delta(t-t').$$

a. Give the associated continuity equation for the conservation of the probability $P(v,t)$ (before averaging over the noise).

b. "Solve" the continuity equation formally (that is, write it as an integral equation). Iterate this equation to express $P(v,t)$, in a formal expansion of $P(v,0)$ and $R(t)$, as a polynomial to the second order in $R(t)$.

c. Derive the associated Fokker-Planck equation obeyed by the noise-averaged probability $\langle P(v,t) \rangle$.

a. $\frac{\partial P}{\partial t} + \frac{\partial(\dot{v}P)}{\partial v} = 0$ *where* $\dot{v} = \frac{dv}{dt} = -\gamma v + f + R$

b. The formal solution is

$$P(v,t) = e^{-t\hat{L}}P(v,0) - \int_0^t d\tau e^{-(t-\tau)\hat{L}} \frac{\partial[R(\tau)e^{-\tau\hat{L}}P(v,0)]}{\partial v}$$

$$+ \int_0^t d\tau e^{-(t-\tau)\hat{L}} \frac{\partial}{\partial v}\left[R(\tau) \int_0^\tau d\tau' e^{-(\tau-\tau')\hat{L}} \frac{\partial\{R(\tau')P(v,\tau')\}}{\partial v} \right]$$

where the operator is defined by $\hat{L}P \equiv \frac{\partial}{\partial v}\left[(-\gamma v + f)P\right]$.

c. Average over noise, we get

$$\frac{\partial\langle P \rangle}{\partial t} = -\frac{\partial}{\partial v}\left[(-\gamma v + f)P\right] + \frac{C}{2}\frac{\partial^2 P}{\partial v^2}.$$

[JSW]

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